Double and Half angle Formulas

Jeff Hicks

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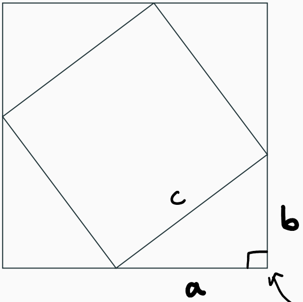
UC Berkeley

Outline

- Recap:Law of Cosines
- Double Angle Formulas
- Half Angle Formulas

Law of Cosines

Review: Pythagorean Theorem

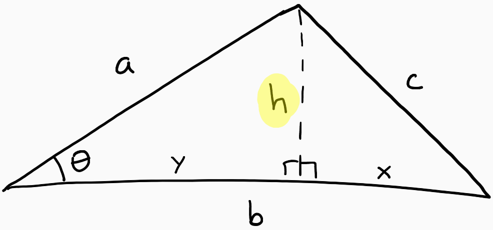


$$a^2+b^2=c^2$$

$$c=\sqrt{a^2+b^2}$$

What happens if this is not a right angle

Computing all side lengths from SAS



Q: What is c?

$c^2 = h^2 + x^2$ (if I knew h & x)

$h = a \cdot \sin\theta$

$x = b - y$

$y = a \cdot \cos\theta$

So: I ~~know~~ know x, h, so!

$c^2 = (a \cdot \sin\theta)^2 + (b - a\cos\theta)^2$ (we could stop!)

For convenience:

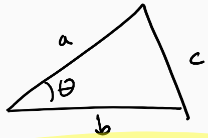
$c^2 = a^2 \cdot (\sin\theta)^2 + b^2 - 2ab\cos\theta + a^2(\cos\theta)^2$

$c^2 = a^2((\sin\theta)^2 + (\cos\theta)^2) + b^2 - 2ab\cos\theta$

$= a^2 + b^2 - 2ab\cos\theta.$

The Law of Cosines

For any triangle



$$\boxed{c^2 = a^2 + b^2 - 2ab\cos\theta.}$$

Comparing Pythagorean to Law of Cosines

Original $\quad c^2 = a^2 + b^2$

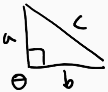
New $\quad c^2 = a^2 + b^2 - 2ab\cos\theta$

$\theta = 0 \quad c^2 = a^2 + b^2 - 2ab$

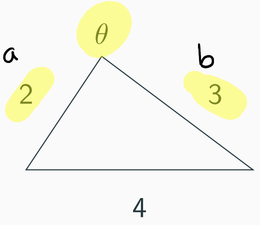
$c^2 = (a-b)^2$ or $(b-a)^2$

And on right, $c = a - b$. So, we're good.

$\theta = \frac{\pi}{2} \quad c^2 = a^2 + b^2$



Example: Compute $\cos(\theta)$ of the below triangle.



By Law of cosines

$$(4)^2 = (2)^2 + (3)^2 - 2(\overset{a}{2}\cdot\overset{b}{3})\cos\theta$$

$$16 = 4 + 9 - 12\cos\theta$$

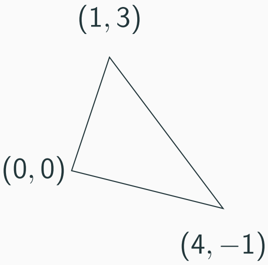
$$3 = -12\cos\theta$$

$$-\frac{1}{4} = \boxed{-\frac{3}{12} = \cos\theta.}$$

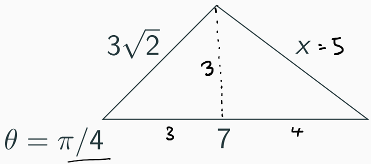
Now we could solve for θ using $\cos^{-1}$.

(SAS type identity)

Example: Compute $\cos(\theta)$ at the origin.



Finding the Missing Side Length



$$x^2 = (3\sqrt{2})^2 + 7^2 - 2(3\sqrt{2}\cdot 7)\cos\theta$$

$$\cos\theta = \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}.$$

$$x^2 = (3\sqrt{2})^2 + 7^2 - 2\cdot 3\cdot\sqrt{2}\cdot 7\cdot\frac{1}{\sqrt{2}}$$

$$x^2 = 9\cdot 2 + 49 - 6\cdot 7$$

$$x^2 = 18 + 49 - 42$$

$$x^2 = 18 + 7$$

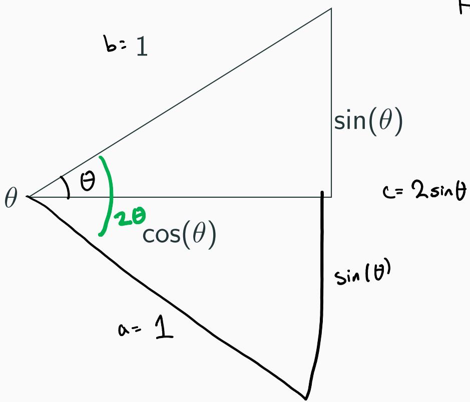
$$x^2 = 25 \qquad x = 5.$$

When to Use law of Cosines

Takeaways: Comparison between right triangles and general triangles

Double Angle Formulas

Goal: compute $\cos(2\theta)$ from $\cos(\theta)$.



For very big Δ, Law of cosines:

$$c^2 = a^2 + b^2 - 2ab\cos\theta$$

$$(2\sin\theta)^2 = 1^2 + 1^2 - 2\cdot 1\cdot 1\cdot\cos(2\theta)$$

$$-2\cos(2\theta) = (2\sin\theta)^2 - 2$$

$$\cos 2\theta = -2(\sin\theta)^2 + 1.$$

$$\cos 2\theta = -2(1-(\cos\theta)^2) + 1$$

$$= \boxed{2(\cos\theta)^2 - 1}.$$

Identity

$$\cos(2\theta) = 1 - 2(\sin(\theta))^2$$

$$\cos(2\theta) = 2(\cos(\theta))^2 - 1$$

$$\cos(2\theta) = (\cos(\theta))^2 - (\sin(\theta))^2$$

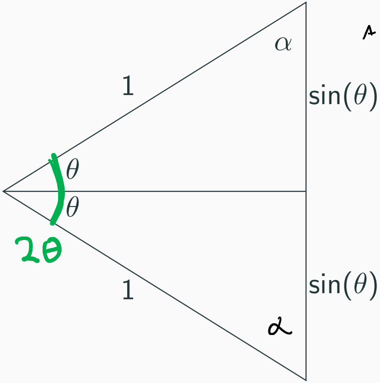
$\cos^2\theta + \sin^2\theta = 1$

Example

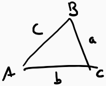
Given that $\cos(\theta) = \frac{1}{3}$, compute $\cos(2\theta)$.

$$\begin{aligned}\cos(2\theta) &= 2\cdot(\cos\theta)^2 - 1\\ &= 2\cdot\left(\tfrac{1}{3}\right)^2 - 1\\ &= 2/9 - 1 = -7/9.\end{aligned}$$

Calculating $\sin(2\theta)$ from $\sin(\theta)$



Law of sines



$$\frac{\sin(A)}{a} = \frac{\sin(B)}{b} = \frac{\sin(C)}{c}$$

$$\frac{\sin(2\theta)}{\sin\theta + \sin\theta} = \frac{\sin(\alpha)}{1}.$$

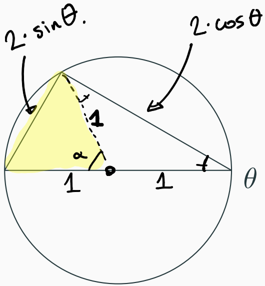
What is α? $\quad \alpha + \alpha + 2\theta = \pi$

$$\alpha = \frac{\pi}{2} - \theta.$$

$$\frac{\sin(2\theta)}{2\cdot\sin(\theta)} = \frac{\sin\left(\frac{\pi}{2} - \theta\right)}{1}.$$

$$\boxed{\sin(2\theta) = 2\sin\theta\cos\theta.}$$

Another Method



$\sin(2\theta) = 2\sin\theta\cos\theta$.

Area of big Δ is

$\frac{1}{2}\cdot b\cdot h = \frac{1}{2}\cdot 2\cos\theta\cdot 2\sin\theta$

$= 2\cos\theta\sin\theta$.

Area of yellow Δ is

$\frac{1}{2}\cdot 1\cdot 1\cdot\sin(\alpha) = \frac{1}{2}\sin(\alpha)$.

$\alpha = 2\theta$.

$\frac{1}{2}$ Area of big Δ = Area of yellow $\Delta = \frac{1}{2}\sin(2\theta)$.

$\frac{1}{2}\cdot 2\cos\theta\cdot\sin\theta = \frac{1}{2}\sin(2\theta)$

Double Angle Formula for Sine

Identity

$$\sin(2\theta) = 2\cos(\theta)\sin(\theta)$$

Computing Tangent

$$\sin(2\theta) = 2\cos(\theta)\sin(\theta)$$
$$\cos(2\theta) = (\cos(\theta))^2 - (\sin(\theta))^2$$

$$\tan(2\theta) = \frac{\sin(2\theta)}{\cos(2\theta)} = \frac{2\cos(\theta)\sin(\theta)}{(\cos(\theta))^2 - (\sin\theta)^2}.$$

Divide both top & bot by $(\cos\theta)^2$

$$= \frac{2 \cdot \frac{\sin(\theta)}{\cos(\theta)}}{\frac{\cos(\theta)^2}{(\cos\theta)^2} - \frac{(\sin\theta)^2}{(\cos\theta)^2}} = \frac{2\tan\theta}{1-(\tan\theta)^2}.$$

Double Angle Formula for Tangent

Identity

$$\tan(2\theta) = \frac{2\tan(\theta)}{1 - (\tan(\theta))^2}$$

Example

Suppose that $\tan(\theta) = 3$. Compute $\tan(2\theta)$.

$$\tan(2\theta) = \frac{2 \cdot 3}{1-(3)^2} = \frac{2 \cdot 3}{1-9} = \frac{2 \cdot 3}{-8} = -\frac{3}{4}.$$

Half Angle Formulas

Computing $\sin(\theta/2)$

$$\cos(2\alpha) = 1 - 2(\sin^2(\alpha))$$

$$\cos(2\alpha) - 1 = -2(\sin(\alpha))^2$$

$$\frac{1-\cos(2\alpha)}{2} = (\sin(\alpha))^2$$

$$\sin\alpha = \pm\sqrt{\frac{1-\cos(2\alpha)}{2}} \qquad \alpha = \theta/2$$

$$\sin\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos(\theta)}{2}}.$$

Computing $\cos(\theta/2)$

$$\cos(2\alpha) = 2(\cos(\alpha))^2 - 1$$

Computing $\tan(\theta/2)$

$\tan(\alpha) = \frac{\sin(\theta)}{\cos(\theta)} = \frac{2\sin(\theta)\cos(\theta)}{2\cos^2(\theta)}$

Half Angle Formulas

Identity

$$\cos\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1+\cos(\theta)}{2}}$$
$$\cos\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1-\cos(\theta)}{2}}$$
$$\tan\left(\frac{\theta}{2}\right) = \frac{\sin(\theta)}{1+\cos(\theta)}$$