Law of Sines and Law of Cosines

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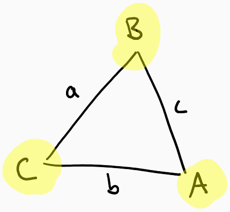
Outline

- Law of Sines
- Law of Cosines

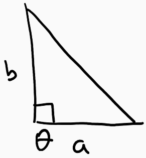
Review: Area of a Triangle



Recall: Area is $\frac{1}{2}\, a \cdot b \cdot \sin(C)$

→ Previously only able to look at areas of right triangles

$\theta = \frac{\pi}{2}$ $\quad\quad \sin(\theta) = 1$

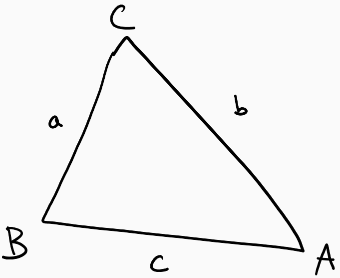


Area: $\frac{1}{2}\, a \cdot b$

Theme: Extends facts about right triangles to all triangles

Notation: a, b, c be sidelengths
$A\ B\ C$ be opposite angles

Area of a triangle in three ways



$$\boxed{\text{Area} = \frac{1}{2}\, a \cdot b \cdot \sin(C)}$$

$$\text{Area} = \frac{1}{2}\, b \cdot c \cdot \sin(A)$$

$$\text{Area} = \frac{1}{2}\, a \cdot c \cdot \sin(B)$$

We write instead $\dfrac{\text{Area} \cdot 2}{a \cdot b \cdot c}$

$$\frac{2\,\text{Area}}{a \cdot b \cdot c} = \frac{2 \cdot \frac{1}{2}\, a \cdot b \cdot \sin(C)}{a \cdot b \cdot c} = \boxed{\frac{\sin(C)}{c} = \frac{\sin(A)}{a} = \frac{\sin(B)}{b}}$$

The law of Sines

Identity: Given a triangle with sides a, b, c,
and opposite angles A, B, C,

We have the equality

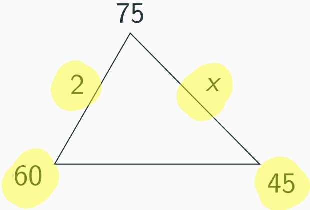
$$\frac{\sin(A)}{a} = \frac{\sin(B)}{b} = \frac{\sin(C)}{c}$$

Example:Find the measure of the side x

Is it possible to answer?

We have 3 angles (AAA not enough)

1 side (ASA, totally determine $\triangle$)



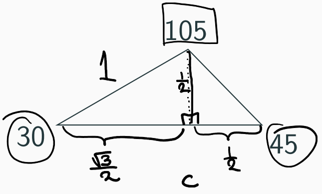
Law of sines

$$\frac{\sin\left(\frac{\pi}{4}\right)}{2} = \frac{\sin\left(\frac{\pi}{3}\right)}{x}$$

$$\frac{\frac{\sqrt{2}}{2}}{2} = \frac{\sqrt{3}/2}{x}$$

$$\boxed{x = \frac{\sqrt{3}}{\sqrt{2}} \cdot 2 = \sqrt{3}\cdot\sqrt{2} = \sqrt{6}}$$

Example: Compute sin(105).



Start by fixing some side length.

to compute sin(105)

Maybe I want to compute the length of c

because

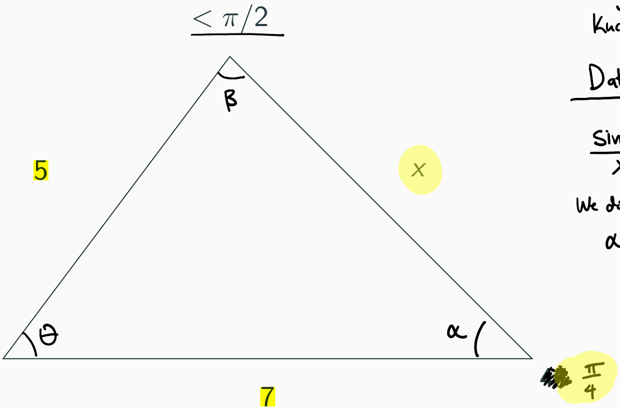
$$\frac{\sin(105)}{c} = \frac{\sin(45)}{1}$$

From 30-60-90 and 45-45-90 right △ rules

$$c = \frac{\sqrt{3}}{2} + \frac{1}{2} = \frac{2+\sqrt{3}}{2}.$$

$$\left.\frac{\sin(105)}{\left(\frac{2+\sqrt{3}}{2}\right)} = \frac{\sqrt{2}}{2}\right\} \quad \sin(105) = \frac{\sqrt{2}}{2+\sqrt{3}}$$

Example: Find x



Why do we need to know the top ∡ is $< \pi/2$.

Data given is SSA.

$$\frac{\sin(\theta)}{x} = \frac{\sin(\frac{\pi}{4})}{5}$$

We don't know θ!

$$\alpha + \beta + \theta = \pi$$

$$\theta = \pi - \alpha - \beta = \pi - \frac{\pi}{4} - \beta$$

$$\theta = \frac{\pi}{2} - \beta.$$

$$\frac{\sin(\beta)}{7} = \frac{\sin(\frac{\pi}{4})}{5}.$$

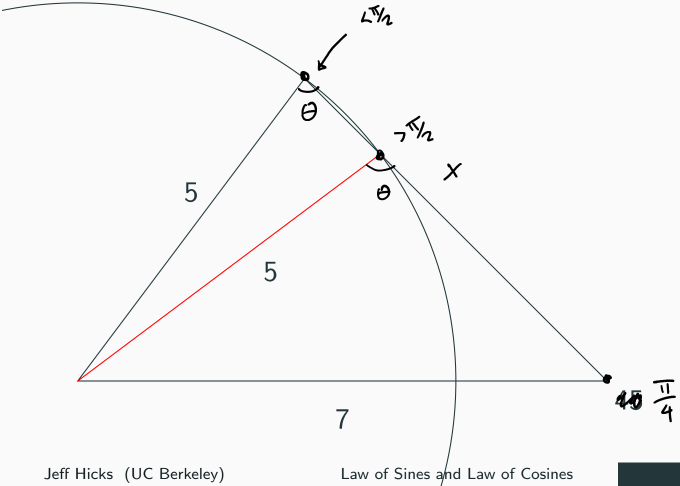
$$\sin(\beta) = \frac{\sqrt{2}\cdot 7}{2\cdot 5} = \frac{7}{10}\sqrt{2}$$

$$\beta = \sin^{-1}\left(\frac{7}{10}\sqrt{2}\right)$$

$$\boxed{\theta = \frac{\pi}{2} - \sin^{-1}\left(\frac{7}{10}\sqrt{2}\right) \qquad \frac{\sin(\theta)}{x} = \frac{\sin(45)}{5} \qquad x = \frac{\sin(45)}{5} \cdot \sin\left(\frac{\pi}{2} - \sin^{-1}\left(\frac{7}{10}\sqrt{2}\right)\right)}$$

There is no SSA Rule!

Why did we need the $< \pi/2$ in the last example?

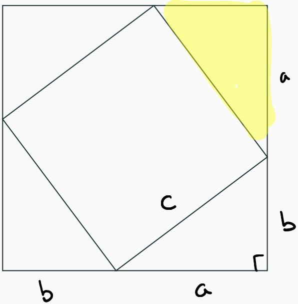


What kinds of problems can this solve?

- Two Angles and a Side.
- Two Sides, and an angle not belonging to both sides (With some additional Information)

Law of Cosines

Review: Pythagorean Theorem



Area of this big square

$$\text{Area} = (a+b)^2 = a^2 + 2ab + b^2$$

$$\text{Area} = c^2 + 4 \cdot \tfrac{1}{2} a \cdot b$$

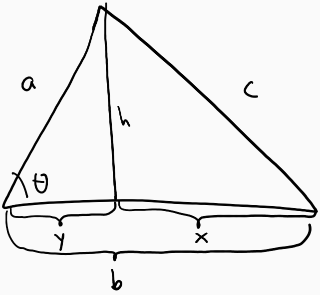
$$= c^2 + 2ab$$

$$a^2 + b^2 + 2ab = c^2 + 2ab$$

$$\underline{a^2 + b^2 = c^2.}$$

Keep in mind

Computing all side lengths from SAS



Notice that $c^2 = x^2 + h^2$

$y = a\cos\theta$

$x = b - a\cos\theta$

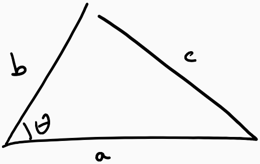
$h^2 = a^2 - y^2$

$$c^2 = (b - a\cos\theta)^2 + a^2 - y^2$$
$$= b^2 - 2ab\cos\theta - a^2\cos^2\theta + a^2 - y^2$$
$$= b^2 + a^2 - 2ab\cos\theta$$

$$a^2 + b^2 = c^2 + 2ab\cos\theta$$

The Law of Cosines

Law of Cosines



$$a^2+b^2 = c^2 + \underbrace{2ab\cos\theta}$$

additional term.
which didn't have for
the right triangle.

Notice when $\frac{\pi}{2} = \theta$ then $\cos\theta = 0$. so,

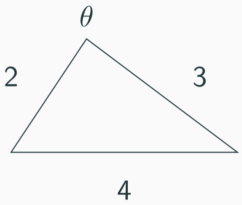
$$a^2+b^2=c^2.$$

as θ goes to 0, then $\cos\theta \to 1$

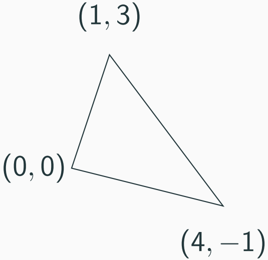
$$a^2+b^2 = c^2+2ab\cos\theta.$$

Comparing Pythagorean to Law of Cosines

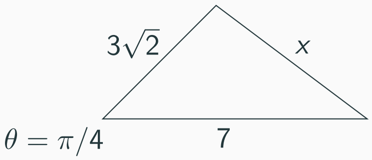
Example: Compute $\cos(\theta)$ of the below triangle.



Example: Compute $\cos(\theta)$ at the origin.



Finding the Missing Side Length



When to Use law of Cosines

Takeaways: Comparison between right triangles and general triangles