

Areas and Approximations

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Apr. 17, 2019

UC Berkeley

Outline

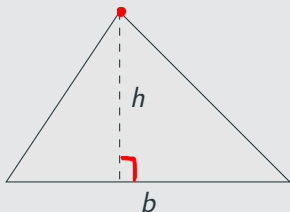
- Computing Areas of Triangles
- Computing Areas of other polygons
- Approximations of $\sin(\theta)$ and $\cos(\theta)$ for small θ .
- Calculating distances

Area of a Triangle

Computing the Area of a Triangle

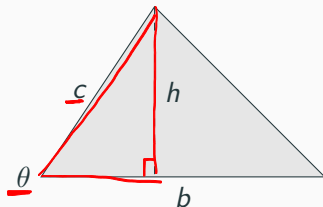
Formula

A triangle has area $\frac{b \cdot h}{2}$ where b is the length of one of the sides (the “base”), and h is the height of the triangle as measured by an altitude.



How can we use trig to help

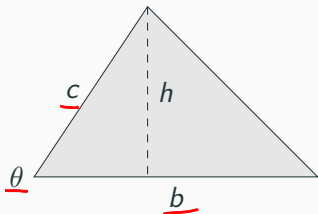
Let's take a closer look at this triangle:



*h is opposite
side of this
smaller triangle*

How can we use trig to help

Let's take a closer look at this triangle:



We see that the height $h = c \cdot \sin(\theta)$. This makes the area

$$\frac{1}{2} b \cdot h = \frac{1}{2} b \cdot c \cdot \sin(\theta)$$

Notice that the area formula does not contain the height of the triangle.

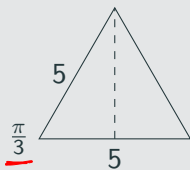
Formula

The area of a triangle with sides b and c meeting at an angle θ is:

$$A = \frac{1}{2} b \cdot c \cdot \sin(\theta)$$

Example

Find the area of an equilateral triangle which has a side length of 5.



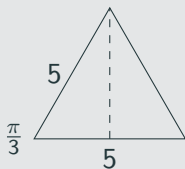
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The area of a triangle with sides b and c meeting at an angle θ is:

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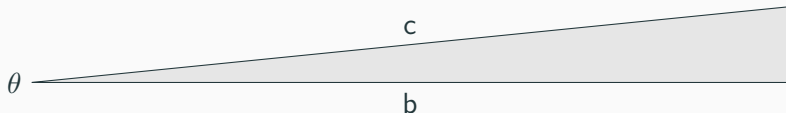
Find the area of an equilateral triangle which has a side length of 5.



$$\begin{aligned} \frac{1}{2}b \cdot c \sin(\theta) &= \frac{1}{2} \cdot 5 \cdot 5 \cdot \sin(\pi/3) \\ &= \frac{1}{2} \cdot 25 \cdot \frac{\sqrt{3}}{2} \\ &= \frac{25\sqrt{3}}{4}. \end{aligned}$$

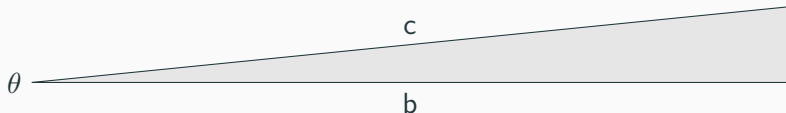
Some other observations about the rule

When θ is close to 0, our triangle looks like this:



Some other observations about the rule

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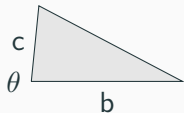


Notice that even if b and c are large, the area of this triangle will be small. This is because when $\theta \sim 0$, then $\sin(\theta) \sim 0$.

$$\frac{1}{2}b \cdot c \cdot \sin(\theta) \sim \frac{1}{2}b \cdot c \cdot 0 \sim 0$$

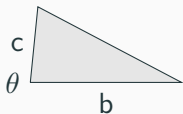
Some other observations about the rule

When θ is close to $\pi/2$, our triangle looks like this:



Some other observations about the rule

When θ is close to $\pi/2$, our triangle looks like this:



When $\theta \sim \overset{\pi/2}{\theta}$, then $\sin(\theta) \sim 1$.

$$\frac{1}{2}b \cdot c \cdot \sin(\theta) \sim \frac{1}{2}b \cdot c$$

which looks a lot like the formula for the area of a right triangle.

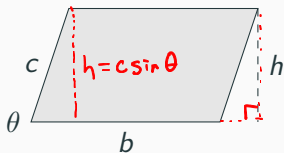
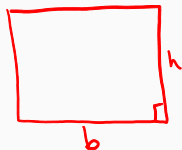
Some Other observations about the rule

The formula

$$\frac{1}{2}b \cdot c \sin(\theta)$$

treats the variables b and c on equal footing.

What about other shapes?



The area of a parallelogram is

$$A = b \cdot h$$

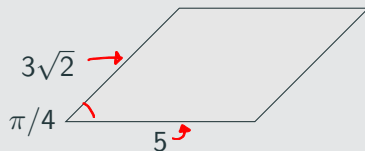
Formula

The area of a parallelogram with sides b and c meeting at an angle θ is:

$$A = b \cdot c \cdot \sin(\theta)$$

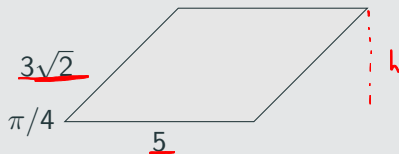
Example

What is the area of the following parallelogram



Example

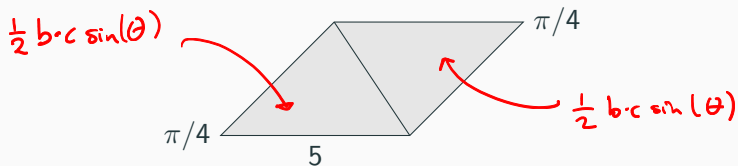
What is the area of the following parallelogram



$$\begin{aligned}
 A &= b \cdot c \cdot \sin(\theta) \\
 &= 5 \cdot 3\sqrt{2} \cdot \sin(\pi/4) \\
 &= 5 \cdot 3\sqrt{2} \cdot \frac{\sqrt{2}}{2} \\
 &= 15.
 \end{aligned}$$

Relation between Parallelograms and Triangles

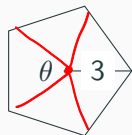
We could have also could have gotten this formula from cutting the parallelogram into two .



This is a trick we can use to find the area of many different shapes.

Area of a regular Pentagon

What is the area of a regular pentagon when the distance from the center to the vertex is 3?

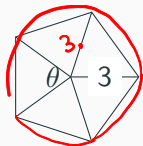


Area of a regular Pentagon

What is the area of a regular pentagon when the distance from the center to the vertex is 3?

The area of each triangle is

$$\frac{1}{2} \cdot 3 \cdot 3 \sin(\theta)$$



The angle $\theta = \frac{2\pi}{5}$. This gives us that the area of the whole pentagon is

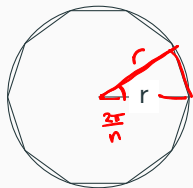
$$\begin{aligned}
 & 5 \cdot \left(\frac{1}{2} \cdot 3 \cdot 3 \sin\left(\frac{2\pi}{5}\right) \right) \\
 &= \frac{45}{3} \left((\sqrt{10} + 2\sqrt{5}) / 4 \right)
 \end{aligned}$$

Handwritten annotations:
 - "5 triangles" with an arrow pointing to the coefficient 5.
 - "Area of each triangle" with an arrow pointing to the expression inside the large parentheses.
 - A red arrow points from the sine term to the final expression.

Area of a n -gon

Repeating on a shape with n equal sides and radius r ,
Area is

$$\begin{aligned}
 & n \cdot \overbrace{\left(\frac{1}{2} r \cdot r \cdot \sin \left(\frac{2\pi}{n} \right) \right)}^{\text{Area of a } \Delta} \\
 &= \frac{n}{2} \left(\sin \left(\frac{2\pi}{n} \right) \right) r^2
 \end{aligned}$$




This looks suspiciously like the formula for the area of the circle. Notice that as n becomes very large, this better and better approximates the area of a circle.

Estimates for Trig Functions

Comparing circles

Our estimate from the last slide says that when n is very large



$$\frac{n}{2} \left(\sin \left(\frac{2\pi}{n} \right) \right) r^2 \approx \pi \cdot r^2$$

from last slide

where the right hand side is the area of a circle.

Comparing circles

of sides.

Our estimate from the last slide says that when n is very large

$$\frac{n}{2} \left(\sin \left(\frac{2\pi}{n} \right) \right) r^2 \approx \pi \cdot r^2$$

where the right hand side is the area of a circle.

We conclude that when n is very large,

$$\begin{aligned} \frac{n}{2} \left(\sin \left(\frac{2\pi}{n} \right) \right) &\approx \pi \\ \sin \left(\frac{2\pi}{n} \right) &\approx \frac{2\pi}{n} \end{aligned}$$

Letting $\theta = \frac{2\pi}{n}$ when n is very big...

Identity

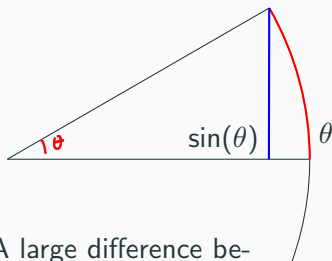
When θ is very small, $\sin(\theta) \approx \theta$.

Remark

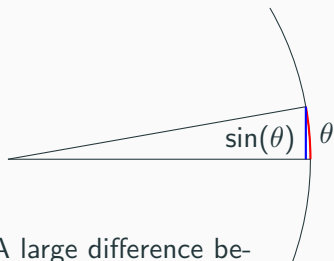
This only works if θ is measured in radians! Fun Fact: This is the real reason why we use radians!

Identity

When θ is very small, $\sin(\theta) \approx \theta$.

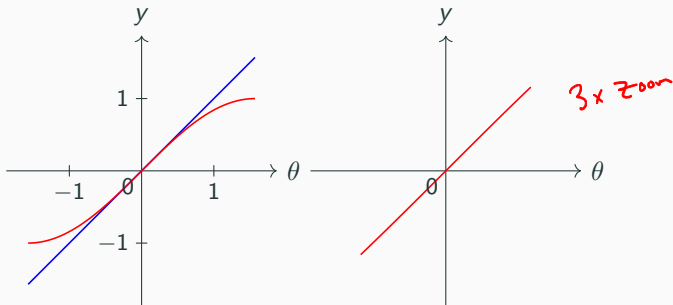


A large difference between $\sin(\theta)$ and θ



A large difference between $\sin(\theta)$ and θ

In red, we have the graph of $y = \sin(\theta)$, while in blue we have the graph of $y = \theta$.



Example

~~Approximate~~ the value of $5 \cdot \sin(.01)$.

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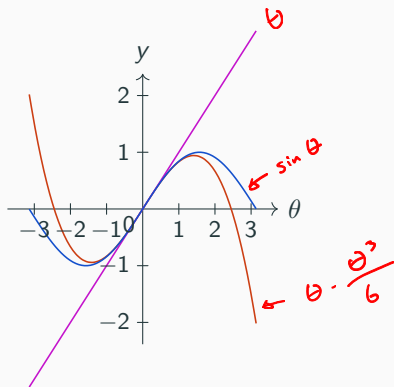
Given that $\sin(\theta) = .02$, approximate θ .

θ is approximately .02.

In fact, there are better and better approximations.

Fact

$$\sin(\theta) = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \dots$$

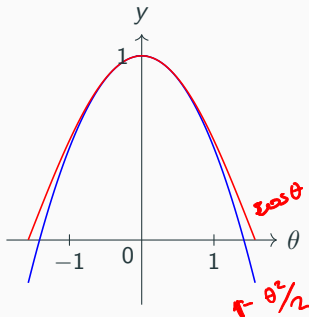


A similar statement is true of $\cos(\theta)$.

Identity

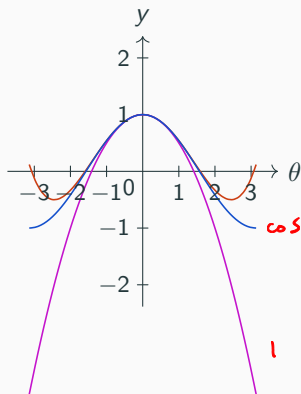
When θ is very small, $\cos(\theta) \approx 1 - \frac{\theta^2}{2}$.

In red, we have the graph of $y = \cos(\theta)$, while in blue we have the graph of $y = 1 - \theta^2/2$.



Fact

$$\cos(\theta) = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \dots$$



For Beyond this Course

In almost every field of quantitative study, one desires to study some function in terms of simpler functions. In this class so far we've seen:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$$

$$\cos(\theta) = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \dots$$

$$\sin(\theta) = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \dots$$

The approximations which have been important to us in this course come from considering just the first few terms of these sums:

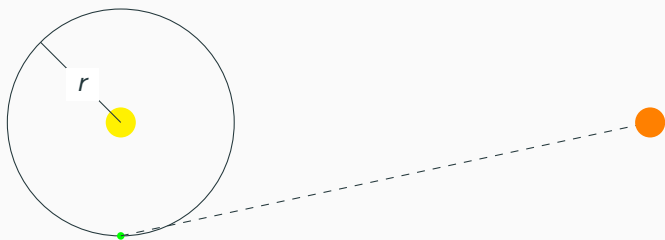
$$e^x \approx 1 + x$$

$$\cos(\theta) \approx 1 - \frac{\theta^2}{2},$$

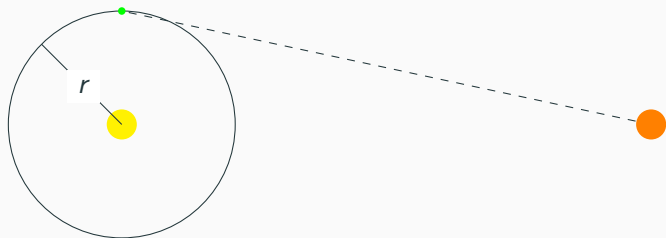
$$\sin(\theta) \approx \theta$$

We call this a “leading order approximation”. One portion of calculus is understanding why we can make these approximations in general situations.

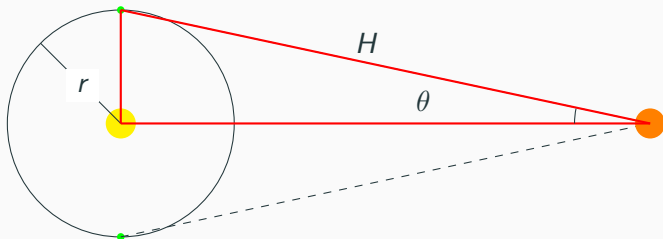
Measuring distance to a star



Measuring distance to a star



Measuring distance to a star



The length H is given by $H = \frac{r}{\sin(\theta)}$.

Since θ is very small, we can approximate $\sin(\theta) \approx \theta$

$$H \approx \frac{r}{\theta}$$

Rule of Thumb for far away things

The distance to something far away is *about*

$$H \approx \frac{r}{\theta}$$

Where θ is the number of radians it moves in your view, and r is the distance between the two points that you make an observation from.

Where else does this occur? In humans, when judging the distance to far away objects! Our depth position is controlled by our abilities to resolve angles.

- Without additional information, humans can tell the distance of a point to about 10 meters.
- Human eyes are about 66mm apart.

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- Without additional information, humans can tell the distance of a point to about 10 meters.
- Human eyes are about 66mm apart.

The angle which a point moves between the left and right eye when at 10 meters is roughly

$$10 \text{ meters} = \frac{33}{\sin(\theta)} \approx \frac{33\text{mm}}{\theta}$$

$$\theta \approx 0.0033 \text{ Radians.}$$

$$\theta \approx 0.0033 \text{ Radians}$$

That angle roughly corresponds to the size of a small beachball at 1 football field away. In fact, humans can resolve angles as small as

0.0003radians.

That's about a small beachball at a kilometer away. This means that there are other difficulties involved in measuring distance (for instance, the relative angle of your eyes to each other.)

Short Survey



<https://forms.gle/3jQ1h2HAJVab1W5H7>