# Areas and Approximations

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# Outline

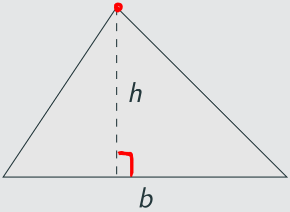
- Computing Areas of Triangles
- Computing Areas of other polygons
- Approximations of $\sin(\theta)$ and $\cos(\theta)$ for small $\theta$.
- Calculating distances

# Area of a Triangle

# Computing the Area of a Triangle

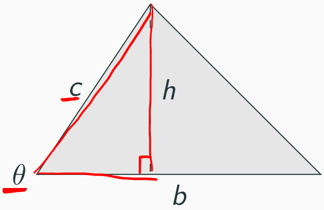
## Formula

*A triangle has area* $\frac{b \cdot h}{2}$ *where* $b$ *is the length of one of the sides (the "base"), and* $h$ *is the height of the triangle as measured by an altitude.*



# How can we use trig to help

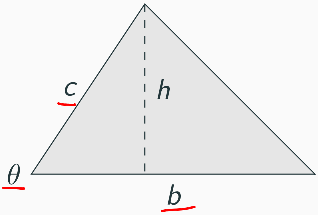
Let's take a closer look at this triangle:




h is opposite
side of this
smaller triangle

# How can we use trig to help

Let's take a closer look at this triangle:




We see that the height $h = c \cdot \sin(\theta)$. This makes the area

$$\frac{1}{2}b \cdot h = \frac{1}{2}b \cdot c \cdot \sin(\theta)$$

Notice that the area formula does not contain the height of the triangle.

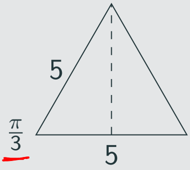
## Formula

*The area of a triangle with sides $b$ and $c$ meeting at an angle $\theta$ is:*

$$A = \frac{1}{2} b \cdot c \cdot \sin(\theta)$$

## Example

Find the area of an equalateral triangle which has a side length of 5.



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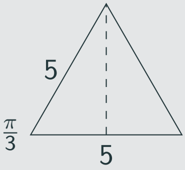
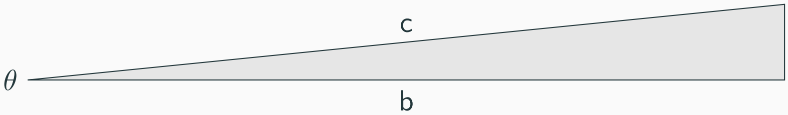



$$\begin{aligned}\frac{1}{2} b \cdot c \sin(\theta) &= \frac{1}{2} \cdot 5 \cdot 5 \cdot \sin(\pi/3) \\ &= \frac{1}{2} \cdot 25 \cdot \frac{\sqrt{3}}{2} \\ &= \frac{25\sqrt{3}}{4}.\end{aligned}$$

# Some other observations about the rule

When $\theta$ is close to 0, our triangle looks like this:



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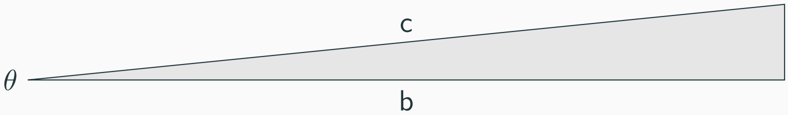
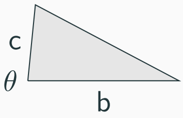



Notice that even if $b$ and $c$ are large, the area of this triangle will be small. This is because when $\theta \sim 0$, then $\sin(\theta) \sim 0$.

$$\frac{1}{2}b \cdot c \cdot \boxed{\sin(\theta)} \sim \frac{1}{2}b \cdot c \cdot 0 \sim 0$$

## Some other observations about the rule

When $\theta$ is close to $\pi/2$, our triangle looks like this:



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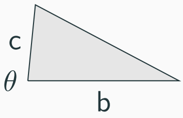



When $\theta \sim \overset{\pi/2}{\cancel{0}}$, then $\sin(\theta) \sim 1$.

$$\frac{1}{2} b \cdot c \cdot \sin(\theta) \sim \frac{1}{2} b \cdot c$$

which looks a lot like the formula for the area of a right triangle.

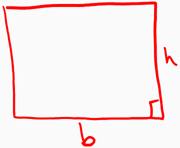
# Some Other observations about the rule

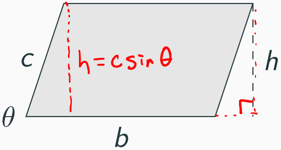
The formula

$$\frac{1}{2}b \cdot c\sin(\theta)$$

treats the variables $b$ and $c$ on equal footing.

## What about other shapes?




The area of a parallelogram is

$$A = b \cdot h$$

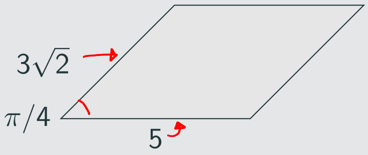
**Formula**

*The area of a parallelogram with sides $b$ and $c$ meeting at an angle $\theta$ is:*

$$A = b \cdot c \cdot \sin(\theta)$$

## Example

What is the area of the following parallelogram



## Example

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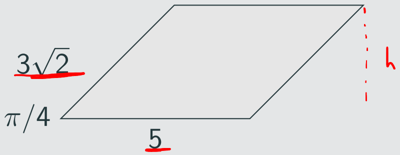
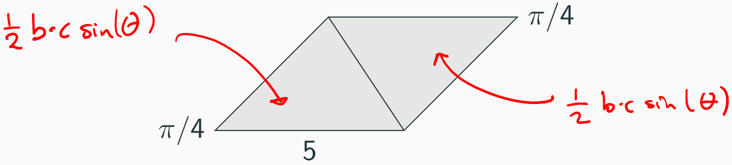



$$
\begin{aligned}
A =& b \cdot c \cdot \sin(\theta) \\
=& 5 \cdot 3\sqrt{2} \cdot \sin(\pi/4) \\
=& 5 \cdot 3\sqrt{2} \cdot \frac{\sqrt{2}}{2} \\
=& 15.
\end{aligned}
$$

## Relation between Parallelograms and Triangles

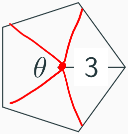
We could have also could have gotten this formula from cutting the parallelogram into two .




This is a trick we can use to find the area of many different shapes.

# Area of a regular Pentagon

What is the area of a regular pentagon when the distance from the center to the vertex is 3?



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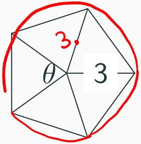



The area of each triangle is

$$\frac{1}{2}3\cdot 3\sin(\theta)$$

The angle $\theta = \frac{2\pi}{5}$. This gives us that the area of the whole pentagon is

Area of each triangle

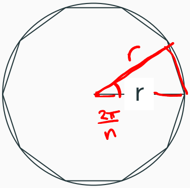
5 triangles

$$5\cdot\left(\frac{1}{2}3\cdot 3\underline{\sin(2\pi/5)}\right)$$

$$=\frac{45}{3}((\sqrt{10}+(2\sqrt{5}))/4)$$

# Area of a $n$-gon

Repeating on a shape with $n$ equal sides and radius $r$,




Area is

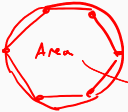
$$n\cdot\left(\overbrace{\frac{1}{2}r\cdot r\cdot\sin\left(\frac{2\pi}{n}\right)}^{\text{Area of a }\Delta}\right)$$
$$=\frac{n}{2}\left(\sin\left(\frac{2\pi}{n}\right)\right)r^2$$

This looks suspiciously like the formula for the area of the circle. Notice that as $n$ becomes very large, this better and better approximates the area of a circle.

# Estimates for Trig Functions

# Comparing circles

Our estimate from the last slide says that when *n* is very large




$$\underbrace{\frac{n}{2}\left(\sin\left(\frac{2\pi}{n}\right)\right) r^2}_{\text{from last slide}} \approx \pi \cdot r^2$$

where the right hand side is the area of a circle.

## Comparing circles

\# of sides.

Our estimate from the last slide says that when $n$ is very large

$$\frac{n}{2}\left(\sin\left(\frac{2\pi}{n}\right)\right) r^2 \approx \pi \cdot r^2$$

where the right hand side is the area of a circle.

We conclude that when $n$ is very large,

$$\begin{aligned}\frac{n}{2}\left(\sin\left(\frac{2\pi}{n}\right)\right) &\approx \pi \\ \sin\left(\frac{2\pi}{n}\right) &\approx \frac{2\pi}{n}\end{aligned}$$

Letting $\theta = \frac{2\pi}{n}$ when $n$ is very big...

**Identity**

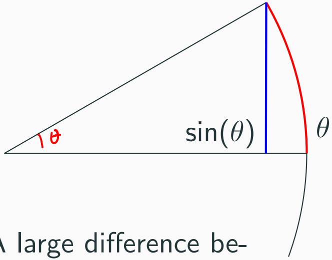
*When $\theta$ is very small,* $\sin(\theta) \approx \theta$.

**Remark**

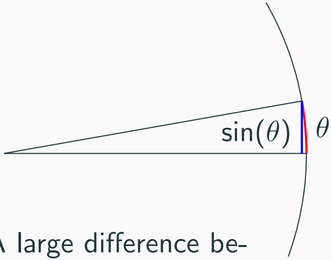
*This only works if $\theta$ is measured in radians! Fun Fact: This is the real reason why we use radians!*

## Identity

*When $\theta$ is very small,* $\sin(\theta) \approx \theta$.

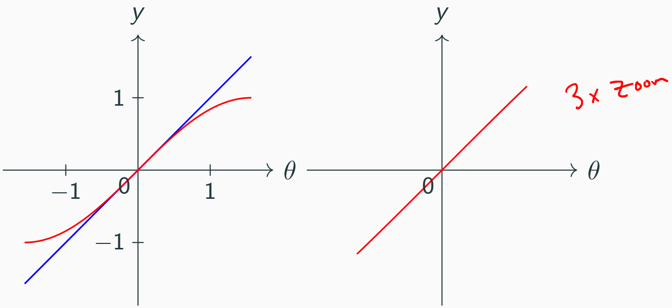



A large difference between $\sin(\theta)$ and $\theta$




A large difference between $\sin(\theta)$ and $\theta$

In red, we have the graph of $y = \sin(\theta)$, while in blue we have the graph of $y = \theta$.



### Example

~~Apprxomate~~ the value of $5 \cdot \sin(.01)$.

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$$5 \cdot \sin(0.1) \approx 5 \cdot (0.1) = .5$$

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## Example

Given that $\sin(\theta) = .02$, approximate $\theta$.

## Example

~~Apprxomate~~ the value of $5 \cdot \sin(.01)$.

$$5 \cdot \sin(0.01) \approx 5 \cdot (0.01) = .05$$

## Example

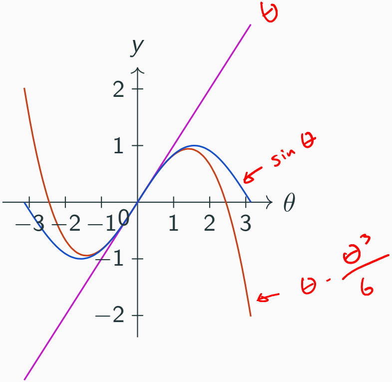
Given that $\sin(\theta) = .02$, approximate $\theta$.

$\theta$ is approximately .02.

In fact, there are better and better approximations.

**Fact**

$\sin(\theta) = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots$

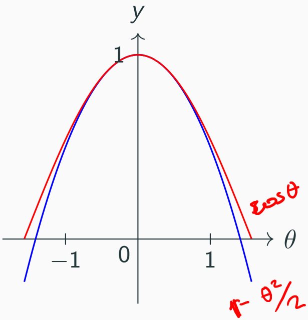


A similar statement is true of $\cos(\theta)$.

**Identity**

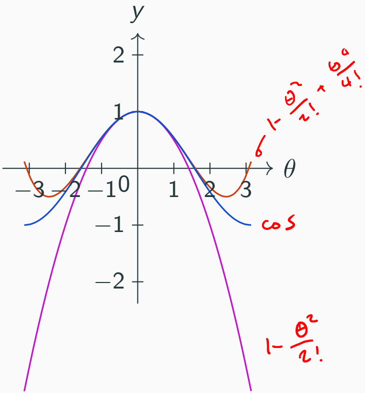
*When $\theta$ is very small,* $\cos(\theta) \approx 1 - \frac{\theta^2}{2}$.

In red, we have the graph of $y = \cos(\theta)$, while in blue we have the graph of $y = 1 - \theta^2/2$.



## Fact

$$\cos(\theta) = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots$$



# For Beyond this Course

In almost every field of quantitative study, one desires to study some function in terms of simpler functions. In this class so far we've seen:

$$\left.\begin{aligned} e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \\ \cos(\theta) &= 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots \\ \sin(\theta) &= \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots \end{aligned}\right\} e^{i\theta} = \cos\theta + i\sin\theta$$

The approximations which have been important to us in this course come from considering just the first few terms of these sums:

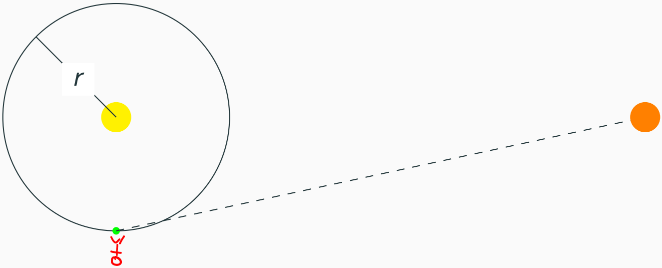
$$e^x \approx 1 + x$$

$$\cos(\theta) \approx 1 - \frac{\theta^2}{2},$$

$$\sin(\theta) \approx \theta$$

We call this a "leading order approximation". One portion of calculus is understanding why we can make these approximations in general situations.

# Measing distance to a star



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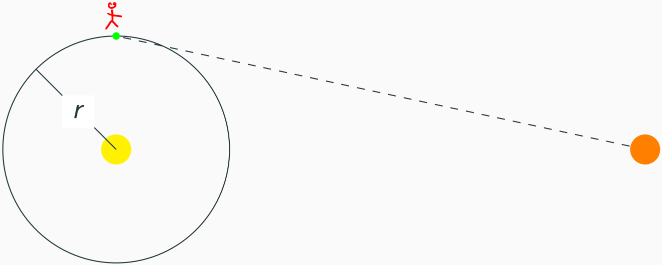
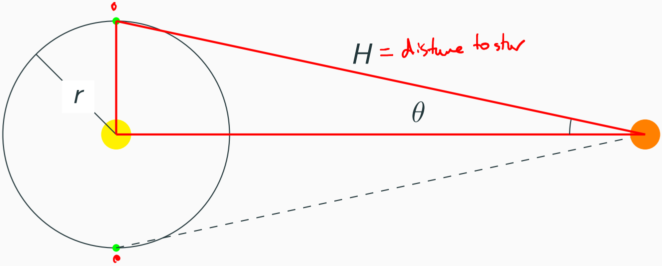


## Measing distance to a star




The length $H$ is given by $H = \frac{r}{\sin(\theta)}$.
Since $\theta$ is very small, we can approximate $\sin(\theta) \approx \theta$

$$H \approx \frac{r}{\theta}$$

# Rule of Thumb for far away things

The distance to something far away is *about*

$$H \approx \frac{r}{\theta}$$

Where $\theta$ is the number of radians it moves in your view, and $r$ is the distance between the two points that you make an observation from.

**Where else does this occur?** In humans, when judging the distance to far away objects! Our depth position is controlled by our abilities to resolve angles.

- Without additional information, humans can tell the distance of a point to about 10 meters.
- Human eyes are about $66mm$ apart.

**Where else does this occur?** In humans, when judging the distance to far away objects! Our depth position is controlled by our abilities to resolve angles.

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- Human eyes are about $66mm$ apart.

The angle which a point moves between the left and right eye when at 10 meters is roughly

$$10 \text{ meters} = \frac{33}{\sin(\theta)} \approx \frac{33mm}{\theta}$$

$$\theta \approx 0.0033 \text{ Radians.}$$

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That angle roughly corresponds to the size of a small beachball at 1 football field away. In fact, humans can resolve angles as small as

$$0.0003 \text{radians}.$$

That's about a small beachball at a kilometer away. This means that there are other difficulties involved in measuring distance (for instance, the relative angle of your eyes to eachother.)

# Short Survey

https://forms.gle/3jQ1h2HAJVab1W5H7