# Identities with Inverse Trig Functions

Jeff Hicks

Apr. 17, 2019

UC Berkeley

## Summary

- Review: “Inverse” trig functions.
- Identies: Compositions of $\sin(\theta)$ and $\sin^{-1}(y)$.
- Example: Inverting functions with terms from trig
- Trig Identities: Right angle Identities
- Trig Identities: Even and Oddness

# Review: Inverse Trig Functions

# Definitions with a bit more meaning

## Definition

The function *arcsine*, which is written as $\sin^{-1}(y)$

- inputs a number $y$ between $-1$ and $1$,
- outputs the angle $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$ whose vertical-coordinate on the unit circle is $y$.

# Definitions with a bit more meaning

**Definition**

The function *arcsine*, which is written as $\sin^{-1}(y)$

- inputs a number $y$ between $-1$ and 1,
- outputs the angle $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$ whose vertical-coordinate on the unit circle is $y$.

**Definition**

The function *arccosine*, which is written as $\cos^{-1}(x)$

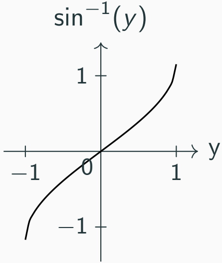
- inputs a number $x$ between 0 and 1,
- outputs the angle $0 \leq \theta \leq \pi$ whose horizontal coordinate on the unit circle is $x$.

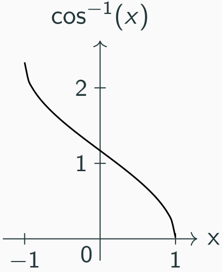
**Definition**

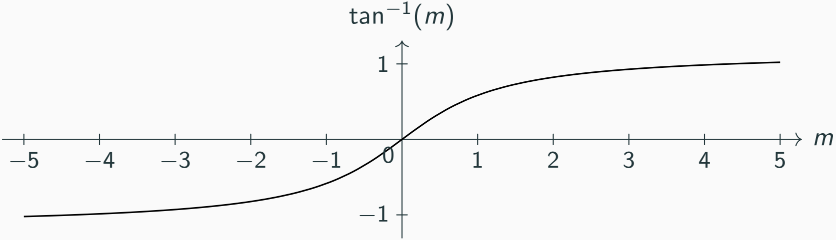
The function *arctangent*, which is written as $\tan^{-1}(y)$

- inputs a number $m$ between $-\infty$ and $\infty$,
- outputs the angle which represents a line of slope $m$.

# What do they look like?



# Cancelling out two functions

## Identity

*For any value in their respective domains,*

$$\sin(\sin^{-1}(y)) = y$$
$$\cos(\cos^{-1}(x)) = x$$
$$\tan(\tan^{-1}(m) = m$$

## Identity

*For any value in their respective domains,*

$$\sin(\sin^{-1}(y)) = y$$
$$\cos(\cos^{-1}(x)) = x$$
$$\tan(\tan^{-1}(m) = m$$

This makes sense! For instance, the first composition asks

$$\sin(\sin^{-1}(y))$$

What is the vertical coordinate of
the angle whose vertical coordinate
is $y$?

# What about the other composition?

$\sin^{-1}(y)$ inverts the function $\sin(\theta)$ for some values of $\theta$. For example:

$$\sin^{-1}(\sin(0)) = \sin^{-1}(0) = 0.$$

$$\sin^{-1}(\sin(\pi/4)) = \sin^{-1}\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}$$

# What about the other composition?

$\sin^{-1}(y)$ inverts the function $\sin(\theta)$ for some values of $\theta$. For example:

$$\sin^{-1}(\sin(0)) = \sin^{-1}(0) = 0.$$

$$\sin^{-1}(\sin(\pi/4)) = \sin^{-1}\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}$$

*But...*

$$\sin^{-1}(\sin(4\pi)) = \sin^{-1}(0) = 0.$$

# Unpacking the definitions again

$$\sin(\sin^{-1}(y))$$

What is the vertical coordinate of
the angle whose vertical coordinate
is $y$?

$$\sin^{-1}(\sin(\theta))$$

What is the angle whose vertical coordinate
matches the vertical coordinate of
the angle $\theta$?

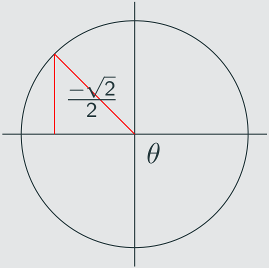
There are many different angles with the same vertical coordinate!

# Computing $\sin^{-1}(\sin(\theta))$

## Example

Evaluate $\sin^{-1}\left(\sin\left(\frac{3\pi}{4}\right)\right)$.

Finding the sine of $3\pi/4$

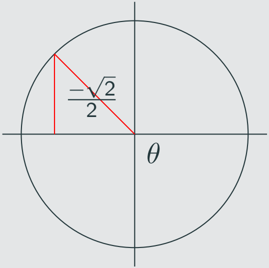


# Computing $\sin^{-1}(\sin(\theta))$

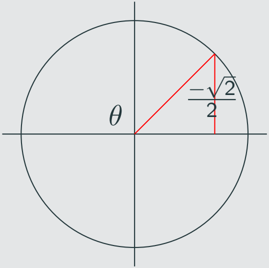
## Example

Evaluate $\sin^{-1}\left(\sin\left(\frac{3\pi}{4}\right)\right)$.

Finding the sine of $3\pi/4$




Finding the angle with height $(\sqrt{2})/2$

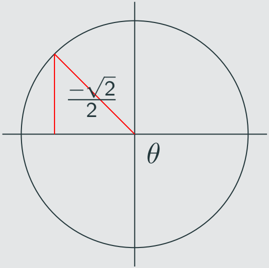


# Computing $\sin^{-1}(\sin(\theta))$

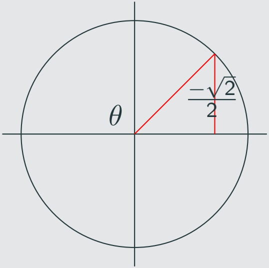
## Example

Evaluate $\sin^{-1}\left(\sin\left(\frac{3\pi}{4}\right)\right)$.

Finding the sine of $3\pi/4$




Finding the angle with height $(\sqrt{2})/2$




In this example, $\sin^{-1}(\sin(3\pi/4)) = \pi/4$.

# Compositions of trig functions and their “inverses”

## Recall

| *Function* | *Domain* | *Range (Outputs)* |
|---|---|---|
| $\sin^{-1}(\theta)$ | $[-1,1]$ | $[-\pi/2,\pi/2]$ |
| $\cos^{-1}(\theta)$ | $[-1,1]$ | $[0,\pi]$ |
| $\tan^{-1}(\theta)$ | $(-\infty,\infty)$ | $(-\pi/2,\pi/2)$ |

## Example

$$\begin{aligned}\sin^{-1}(\sin(5\pi/6)) &= \sin^{-1}(1/2)\\ &= \pi/6\end{aligned}$$

## Recall

| *Function* | *Domain* | *Range (Outputs)* |
|---|---|---|
| $\sin^{-1}(\theta)$ | $[-1, 1]$ | $[-\pi/2, \pi/2]$ |
| $\cos^{-1}(\theta)$ | $[-1, 1]$ | $[0, \pi]$ |
| $\tan^{-1}(\theta)$ | $(-\infty, \infty)$ | $(-\pi/2, \pi/2)$ |

## Example

$$\begin{aligned}\cos^{-1}(\cos(-\pi/3)) &= \cos^{-1}(1/2)\\ &= \pi/3\end{aligned}$$

## Recall

| *Function* | *Domain* | *Range (Outputs)* |
|---|---|---|
| $\sin^{-1}(\theta)$ | $[-1, 1]$ | $[-\pi/2, \pi/2]$ |
| $\cos^{-1}(\theta)$ | $[-1, 1]$ | $[0, \pi]$ |
| $\tan^{-1}(\theta)$ | $(-\infty, \infty)$ | $(-\pi/2, \pi/2)$ |

## Example

$$\begin{aligned} \tan^{-1}(\tan(3\pi/4)) &= \tan^{-1}(-1) \\ &= -\pi/4 \end{aligned}$$

# Drawing Triangles to make computations

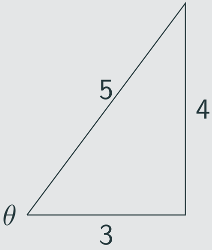
## Example

Compute $\cos\left(\sin^{-1}\left(\frac{4}{5}\right)\right)$.

# Drawing Triangles to make computations

## Example

Compute $\cos\left(\sin^{-1}\left(\frac{4}{5}\right)\right)$.




So what we are asking in this instance is what is $\cos(\theta)$ of the above angle, which is $\frac{3}{5}$.

# Drawing Triangles to make computations

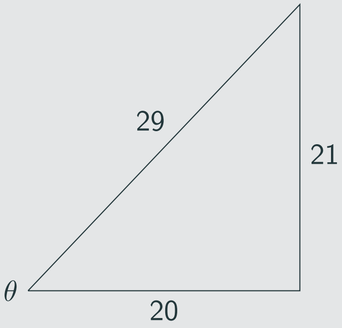
## Example

Compute $\tan\left(\sin^{-1}\left(\frac{21}{29}\right)\right)$.

# Drawing Triangles to make computations

## Example

Compute $\tan\left(\sin^{-1}\left(\frac{21}{29}\right)\right)$.




So in this case we are asking about $\tan(\theta)$ of this angle, which is $\frac{21}{20}$

## Important takeaway

**Order matters!** Keep in mind the difference between these two compositions:

| | |
|---|---|
| $\sin(\sin^{-1}(y)) = y$ | $\sin^{-1}(\sin(\theta))$ may or may not be $\theta$ |
| $\cos(\cos^{-1}(x)) = x$ | $\cos^{-1}(\cos(\theta))$ may or may not be $\theta$ |
| $\tan(\tan^{-1}(m) = m$ | $\tan^{-1}(\tan(\theta))$ may or may not be $\theta$ |

# Identities with $\sin^{-1}$ and $\cos^{-1}$

# Right angle Identities

Recall with $\sin(\theta)$ and $\cos(\theta)$ that we have the following right angle identities:

$$\sin(\theta) = \cos(\theta - \pi/2)$$

$$\cos(\theta) = \sin(\theta + \pi/2)$$

These should translate into right-angle identities for arcsine and arccosine.

# Right angle Identities for $\sin^{-1}$ and $\cos^{-1}$

**Identity**

$\sin^{-1}(y) + \cos^{-1}(y) = \frac{\pi}{2}$.

# Right angle Identities for $\sin^{-1}$ and $\cos^{-1}$

## Identity

$\sin^{-1}(y) + \cos^{-1}(y) = \frac{\pi}{2}$.

## Example

Given that

$$\sin^{-1}\left(\frac{1}{2}\sqrt{2+\sqrt{2}}\right) = \frac{\pi}{8},$$

compute $\cos^{-1}\left(\frac{1}{2}\sqrt{2+\sqrt{2}}\right)$.

# Right angle Identities for $\sin^{-1}$ and $\cos^{-1}$

## Identity

$\sin^{-1}(y) + \cos^{-1}(y) = \frac{\pi}{2}$.

## Example

Given that

$$\sin^{-1}\left(\frac{1}{2}\sqrt{2+\sqrt{2}}\right) = \frac{\pi}{8},$$

compute $\cos^{-1}\left(\frac{1}{2}\sqrt{2+\sqrt{2}}\right)$.

**Solution:**

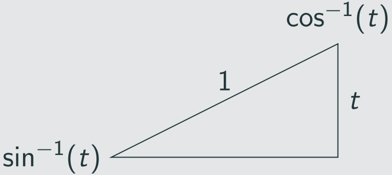
$$\cos^{-1}\left(\frac{1}{2}\sqrt{2+\sqrt{2}}\right) = \frac{\pi}{2} - \frac{\pi}{8} = \frac{3\pi}{8}.$$

# How to remember this identity

# How to remember this identity

**Proof.**

Let's draw a picture which contains both $\sin^{-1}(t)$ and $\cos^{-1}(t)$.




**Important Observation:** The angles of a triangle add up to $\pi$ radians.

$$\sin^{-1}(t) + \cos^{-1}(t) + \frac{\pi}{2} = \pi$$

$$\sin^{-1}(t) + \cos^{-1}(t) = \frac{\pi}{2}$$

# Identities with negation

# Review of Trig identities with negation

## Recall

*From last section we had the following two identities with negation.*

$$-\sin(\theta) = \sin(-\theta)$$
$$-\tan(\theta) = \tan(-\theta).$$

# Review of Trig identities with negation

## Recall

*From last section we had the following two identities with negation.*

$$-\sin(\theta) = \sin(-\theta)$$
$$-\tan(\theta) = \tan(-\theta).$$

## Identity

*Negation effects inverse trig functions in the following way:*

$$\sin^{-1}(-y) = -\sin^{-1}(y)$$
$$\tan^{-1}(-m) = -\tan^{-1}(m).$$

## Example

If $\sin^{-1}(.9) = 1.119$, what is $\sin^{-1}(-.9)$?

## Example

If $\sin^{-1}(.9) = 1.119$, what is $\sin^{-1}(-.9)$?

By our identity it must be $\sin^{-1}(-.9) = -1.119$.

**Example**

If $\sin^{-1}(.9) = 1.119$, what is $\sin^{-1}(-.9)$?

By our identity it must be $\sin^{-1}(-.9) = -1.119$.

**Example**

For which values of $m$ is $\tan^{-1}(m) = \pi/2 + \tan^{-1}(-m)$?

**Example**

If $\sin^{-1}(.9) = 1.119$, what is $\sin^{-1}(-.9)$?

By our identity it must be $\sin^{-1}(-.9) = -1.119$.

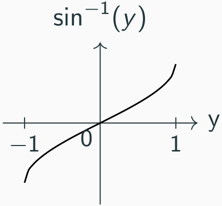
**Example**

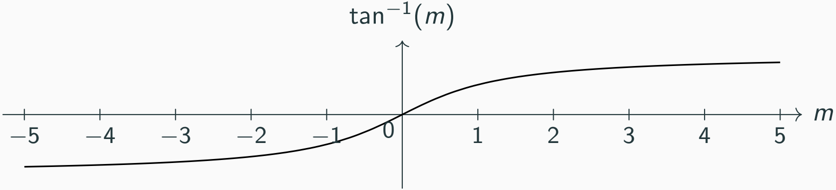
For which values of $m$ is $\tan^{-1}(m) = \pi/2 + \tan^{-1}(-m)$?

By our identity,

$$\begin{aligned}
\tan^{-1}(m) =& \pi/2 + \tan^{-1}(-m) \\
\tan^{-1}(m) =& \pi/2 - \tan^{-1}(m) \\
2\tan^{-1}(m) =& \pi/2 \\
\tan^{-1}(m) =& \frac{\pi}{4} \\
m =& 1
\end{aligned}$$

How can we remember this? Both $\sin^{-1}(y)$ and $\tan^{-1}(m)$ are odd functions.



# What about Arccosine?

Recall that for $\cos(t)$ we have the following identity:

$$-\cos(t) = \cos(t + \pi)$$

This gives us the following identity for $\cos^{-1}(x)$.

**Identity**

$\cos^{-1}(-x) = \pi - \cos^{-1}(x)$

**Example**

Suppose that $\cos^{-1}(x) = 0.7$. Estimate $\cos^{-1}(-x)$.

**Example**

Suppose that $\cos^{-1}(x) = 0.7$. Estimate $\cos^{-1}(-x)$.

$\cos^{-1}(-x) = \pi - \cos^{-1}(x) \sim 3.14 - .7 = 2.44.$

How can we remember this? It can help to draw a picture of $\cos^{-1}(x)$.