# Inverse Trig Functions

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## Summary

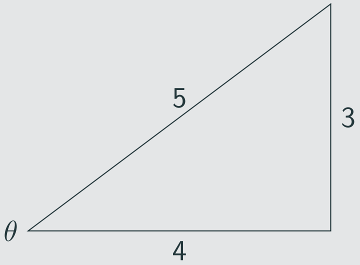
- Question: Why would one want to invert trig functions?
- Review: Inverse functions
- Problem: Why we can't actually invert trig functions.
- Solution: What's the next best thing, and some side effects.

# Background

# Why we want to do this

There are practical reasons to want to invert trigonometric functions.

**Example**




**Question**: What is the measure of $\theta$?

**Idea:** With trigonometry, we know that

$$\sin(\theta) = \frac{3}{5}$$

**Problem:** What does this tell us about $\theta$?

# Re-examination of $\sin(\theta)$

What is $\sin(\theta)$ exactly?

- $\theta$ is the measurement of an angle – in other words, a number.
- $\sin(\theta)$ is a *function*
  - Which inputs a number $\theta$,
  - And outputs a different number (the $y$-coordinate of the point on the unit circle.)

**Goal**

*If we want to learn what $\theta$ is from the function $\sin(\theta)$, we need to find a inverse for the function $\sin(\theta)$.*

# Reviewing Inverse Functions

# Review: Inverses

## Definition

If $f(x)$ is *one-to-one* function, the *inverse* function $f^{-1}(y)$ is

$$f^{-1}(y) = (\text{The number } x \text{ so that } f(x) = y.)$$

*Thataway:* The inverse function $f^{-1}$ is like an “undo” function for $f$,

$$f^{-1}(f(x)) = x$$

## Example

The inverse of $g(x) = 2x + 1$ is $g^{-1}(y) = \frac{y-1}{2}$. We can check this:

$$\begin{aligned} g^{-1}(g(x)) =& \frac{(2x+1)-1)}{2} \\ =& \frac{2x}{2} \\ =& x \end{aligned}$$

## Example

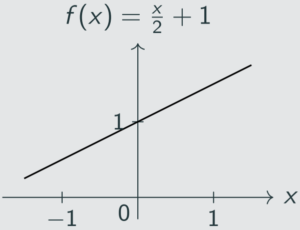
If $f(x) = e^x$, then $f^{-1}(y) = \ln(y)$ because

$$f^{-1}(f(x)) = \ln(e^x) = x.$$

# What about one-to-one.

Recall, that *one-to-one* means that no two inputs produce the same output.

## Example



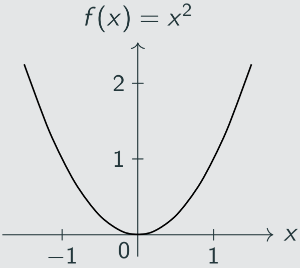

$f(x) = \frac{x}{2} + 1$ is an example of a one-to-one function, whose inverse is

$$f^{-1}(y) = 2(y - 1).$$

# Horizontal Line Test

The horizontal line test says that a function $f(x)$ is one-to-one if every horizontal line only touches the graph of $f(x)$ at a single point.

**Example**




$g(x) = x^2$ is not an example of a one-to-one function, as both $g(1) = 1$ and $g(-1) = 1$.

Rule: If $f(x)$ fails the horizontal line test it *cannot have an inverse!*

This is important to remember, as we are about to try to work around this rule.

# What if $f$ is not one-to-one?

## Big Idea

*If $f(x)$ fails the horizontal line test, we can still sometimes construct an inverse for a portion of $f$.*

## Example

Consider $f(x) = x^2$. Then $g(y) = \sqrt{y}$ is almost an inverse for $f(x)$, because whenever $x$ is a positive number, we have

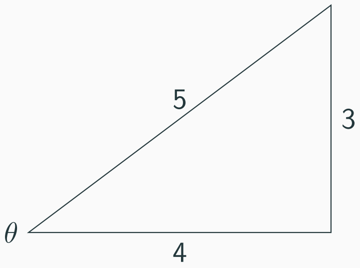
$$g(f(x)) = \sqrt{x^2} = x.$$

But this doesn't work when $x$ is negative. For example.

$$g(f(-1)) = \sqrt{(-1)^2} = \sqrt{(}1) = 1.$$

# Defining $\sin^{-1}(x)$

# Returning to our main problem.




**Question**: What is the measure of $\theta$?

**Idea:** With trigonometry, we know that

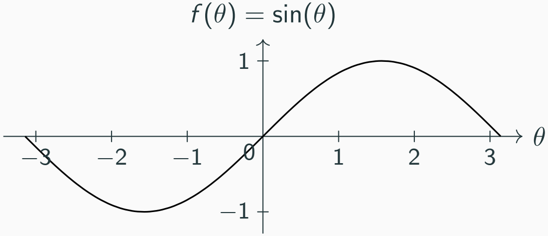
$$\sin(\theta) = \frac{3}{5}$$

If $\sin(\theta)$ has an inverse function, then

$$\theta = \sin^{-1}\left(\frac{3}{5}\right).$$

# Can we invert $\sin(\theta)$?

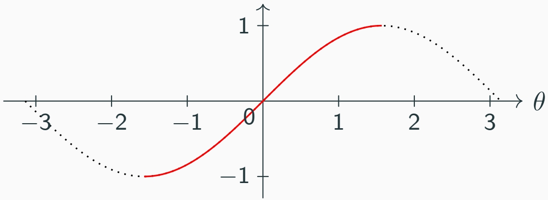
We've set ourselves the goal of inverting $\sin(\theta)$.




The sine function *really, really fails* the horizontal line test.

# Can we invert $\sin(\theta)$?

What if we are a little bit less ambitious, and only try to invert a portion of the function?



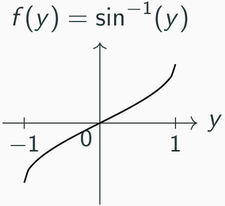

Over this smaller portion, the function has an inverse.

## Definition

The function *arcsine*, which is written as $\sin^{-1}(y)$

- inputs a number $y$ between $-1$ and $1$,
- outputs the angle $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$ with $\sin(\theta) = y$.

The graph of this function looks something like this:



## Is $\sin^{-1}(x)$ an inverse?

$\sin^{-1}(y)$ inverts the function $\sin(\theta)$ for some values of $\theta$. For example

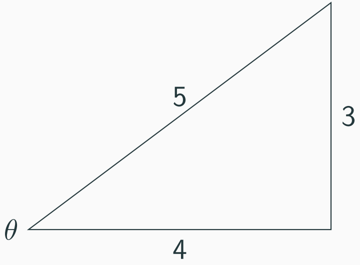
$$\sin^{-1}(\sin(0)) = \sin^{-1}(0) = 0.$$

$$\sin^{-1}(\sin(\pi/4)) = \sin^{-1}\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}$$

*But...*

$$\sin^{-1}(\sin(4\pi)) = \sin^{-1}(0) = 0.$$

# Returning to our Example




**Question**: What is the measure of $\theta$?

**Idea:** With trigonometry, we know that

$$\sin(\theta) = \frac{3}{5}$$

If $\sin(\theta)$ has an inverse function, then

$$\theta = \sin^{-1}\left(\frac{3}{5}\right).$$

$$\sim .643\text{rads}$$

# Other Trig Functions

# What about Cosine?

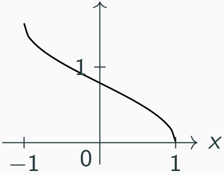
## Definition

The function *arccosine*, which is written as $\cos^{-1}(x)$

- inputs a number $x$ between $-1$ and 1,
- outputs the angle $0 \leq \theta \leq \pi$ with $\cos(\theta) = x$.

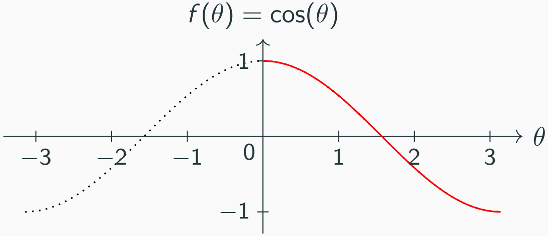
The graph of this function looks something like this:

$f(x) = \cos^{-1}(x)$



# Where is this an inverse for Cosine?

As in the example of $\sin(\theta)$ we only invert a portion of the function.




Notice that the red portion passes the horizontal line test.

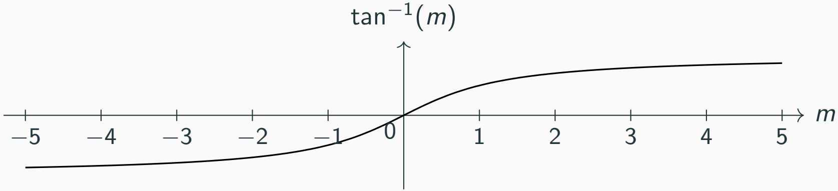
# What about Tangent?

## Definition

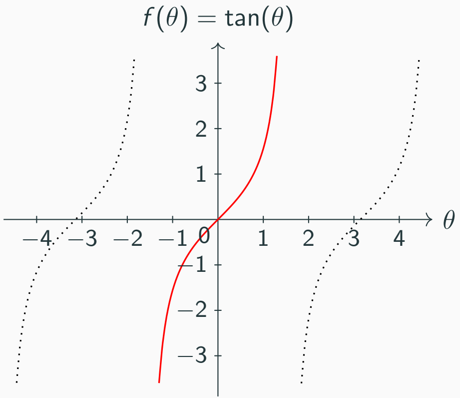
The function *arctangent*, which is written as $\tan^{-1}(m)$

- inputs a number $m$ between $-\infty$ and $\infty$,
- outputs the angle $\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$ with $\tan(\theta) = m$.

The graph of this function looks something like this:



# Where is this an inverse for Tangent?



## Definitions with a bit more meaning

### Definition

The function *arcsine*, which is written as $\sin^{-1}(y)$

- inputs a number $y$ between $-1$ and $1$,
- outputs the angle $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$ whose vertical-coordinate on the unit circle is $y$.

### Definition

The function *arccosine*, which is written as $\cos^{-1}(x)$

- inputs a number $x$ between $0$ and $1$,
- outputs the angle $0 \leq \theta \leq \pi$ whose horizontal coordinate on the unit circle is $x$.

**Definition**

The function *arctangent*, which is written as $\tan^{-1}(y)$

- inputs a number $m$ between $-\infty$ and $\infty$,
- outputs the angle which represents a line of slope $m$.

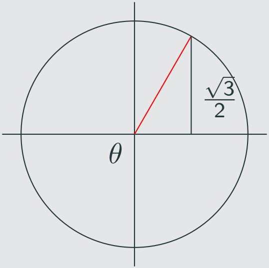
# Examples

# Computing $\sin^{-1}(\theta)$

## Example

Evaluate $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$.

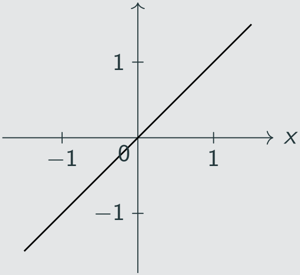
*What is the angle $\theta$ between $-\pi/2$ and $\pi/2$ so that $\sin(\theta) = \frac{\sqrt{3}}{2}$?*




The angle $\theta$ is $\frac{\pi}{3}$.

## Example

$f(x) = x$




What is the angle that the line $y = x$ makes with the $x$-axis?

- The slope of the line is 1.
- $\tan(\theta)$ is the slope of the line.
- Then $\tan^{-1}(1)$ is the measure of the angle.
- $tan^{-1}(1) = \frac{\sqrt{2}}{2}$.

# Domain and Range

# Domain and Range of Inverse Trig Functions

| Function | Domain | Range |
|---|---|---|
| $\sin^{-1}(\theta)$ | $[-1, 1]$ | $[-\pi/2, \pi/2]$ |
| $\cos^{-1}(\theta)$ | $[-1, 1]$ | $[0, \pi]$ |
| $\tan^{-1}(\theta)$ | $(-\infty, \infty)$ | $(-\pi/2, \pi/2)$ |

## Compositions of Inverse Trig Functions

Takeaway from today: Sine and Arcsine cancel each other out in one direction.

$$\sin(\sin^{-1}(y)) = y$$

$$\cos(\cos^{-1}(x)) = x$$

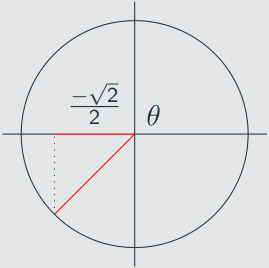
$$\tan(\tan^{-1}(m)) = m$$

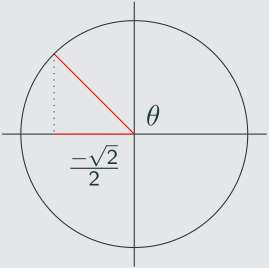
However, the reverse statements *are not true!*

# Computing $\cos^{-1}(\theta)$

## Example

Evaluate $\cos^{-1}\left(\cos\left(\frac{5\pi}{4}\right)\right)$.




However, $\cos^{-1}(y)$ only still takes values between 0 and $\pi$.
This gives $\cos^{-1}\left(\frac{-\sqrt{2}}{2}\right) = \frac{3\pi}{4}$.

We will look at this example in more detail on Wednesday.