PRACTICE MIDTERM PROBLEMS

NAME:

DISCLAIMER: THESE SOLUTIONS MAY HAVE SMALL ERRORS!

**0.1.** Let $S$ be the bottom 5 faces of the cube with corners at $(\pm 1, \pm 1, \pm 1)$.

Compute the flux of the vector field $\langle x, x^2 + y, z^2 \rangle$ through the 5 faces of the cube. (Hint: You do not actually have to integrate over the 5 different faces.)
**Solution:**The trick is to compute the integral of the divergence over the interior, and then subtract of the integral of the flux through the top face.
Let $S_t$ be the top face, and $S_c$ be the entire cube. Since the entire cube is closed, we could instead compute the flux though the entire cube:

$$\begin{aligned}\iint_{S_c} \vec{F} \cdot \vec{n} dS &= \iiint_V \text{div} \vec{F} dV \\ &= \iiint (1 + 1 + 2z) dV \\ &= \iint_{S_c} 2 + 2z dV \\ &= 8 + 0 = 8\end{aligned}$$

Now, we just need to compute the flux through the top face. It comes out to be

$$\iint_{S_t} \vec{F} \cdot \vec{n} dA = \iint_{S_t} \langle x, x^2 + y, 1^2 \rangle \cdot \langle 0, 0, 1 \rangle \, dA = 4$$

So, the flux through our original surface is 16-4=12.

**0.2.** Consider the vector field $\vec{F}(x, y) = \langle 2x \sin^3(y), 3x^2 \sin^2(y) \cos(y) \rangle$. Compute $\int_C \vec{F} \cdot d\vec{r}$ for this vector field along the curve

$$\vec{r}(t) = \langle \cos(t), \sin(t) \rangle$$

as $t$ goes from $-\pi/2$ to $\pi/2$.
**Solution:**This is a conservative vector field, so the path that we take does not matter between 2 points to compute the integral. Instead of taking this half circle, lets take the straight line between $(0, -1)$ and $(0, 1)$. Notice that $x$ is always equal to zero along this curve. Then

$$\int_{C'} \langle 2(0) \sin^3(y), 3(0)^2(y) \cos(y) \rangle \cdot (0, 1) dt = 0$$

**0.3.** Compute the double integral of the function $\iint_R e^{9x^2 + 16y^2} dA$ over the region $R$ where $9x^2 + 16y^2 \leq 1$ and $x \geq 0$.
**Solution:**This is a problem asking you to make a substitution of coordinates. Let's instead use

$$x = r/3 \cos \theta$$

$$y = r/4 \sin \theta$$

Then

$$x_r = \frac{1}{3}\cos(\theta) \qquad x_\theta = \frac{-r}{3}\cos(\theta)$$
$$y_r = \frac{1}{4}\sin(\theta) \qquad y_\theta = \frac{r}{4}\sin(\theta)$$

Then, using the jacobian we have that $dA = \frac{1}{12} r dr d\theta$. Our integral becomes

$$\iint_R e^{9x^2+16y^2} dA = \iint_{r\leq 1} \frac{1}{12} re^r dr d\theta$$

which we can solve by integration by parts.

$$\begin{aligned} &= \frac{1}{6}\pi \int_0^1 re^r dr d\theta \\ &= \frac{1}{6}\pi(re^r - e^r)|_0^1 = \pi/6 \end{aligned}$$

**0.4.** The boundary of a region is given by the curve $r(t) = \left\langle t^2, \sin(t)\right\rangle$ as $t$ goes from $-\pi$ to $\pi$. Compute the area of this region.

**Solution:**I think this is probably one of the more difficult problems that you could be asked. The idea is instead of computing the integral over a region, you want to use Green's theorem in reverse to have an integral over the boundary of the region. So, we look at the vector field $\vec{F} = \langle -y, 0\rangle$

Then we have

$$\begin{aligned} \iint 1 dV &= \iint \operatorname{curl}\vec{F} dV \\ &= \oint \vec{F}\cdot d\vec{r} \\ &= \oint \langle \sin(t), 0\rangle \cdot \langle 2t, \sin(t)\rangle \, dt \\ &= \oint 2t\sin(t) dt \\ &= 2\int_{-\pi}^{\pi} t\sin(t) dt \\ &= 2t\cos(t) + \sin(t)|_0^{2\pi} = 4\pi. \end{aligned}$$

**0.5.** Consider the surface $S$ parameterized by

$$\vec{r}(\theta, z) = \left\langle 1/z^2\cos\theta, 1/z^2\sin\theta, z\right\rangle$$

where $0 \leq 2\pi \leq \theta$ and $1 \leq z$. This surface looks a bit like a spike going off to infinity along the $z$ axis.

- Compute the surface area of the resulting parameterized surface. (Don't Do this, this problem is broken!)
- Consider the vector field
$$\vec{F}(x,y,z) = \frac{1}{2}\langle x, y, 0\rangle .$$
- Compute by hand the flux of the vector field $\vec{F}$ through the surface.
- Let $D$ be disk of unit radius whose center intersects the $z$ axis perpendicularly at $z = 1$. Show the flux of $\vec{F}$ through $D$ is 0.
- Conclude that the volume between $S$ and $D$ is finite.

**Solution:**If we attempt to compute the surface area, we'll need to compute the surface area element $dS$. We have that

$$r_\theta = \left\langle \frac{-1}{z^2}\sin(\theta), \frac{1}{z^2}\cos\theta, 0\right\rangle$$
$$r_z = \left\langle \frac{-2}{z^3}\cos\theta, \frac{-2}{z^3}\sin\theta, 1\right\rangle$$

$$r_\theta \times r_z = \left\langle \frac{1}{z^2}\cos\theta, \frac{1}{z^2}\sin\theta, \frac{2}{z^5} \right\rangle$$

Then length of this is

$$\sqrt{1/z^4 - 2/z^{10}}dzd\theta$$

which simplifies to

$$1/z^2\sqrt{1 - 2/z^6}dzd\theta$$

Which isn't very good! Sorry, bad problem.
Ok, now let's look at computing the volume instead. We have that

$$\vec{F} \cdot \vec{n}dS = \frac{1}{2}\left\langle \frac{1}{z^2}\cos\theta, \frac{1}{z^2}\sin\theta, 0 \right\rangle \cdot \left\langle \frac{1}{z^2}\cos\theta, \frac{1}{z^2}\sin\theta, \frac{2}{z^5} \right\rangle$$

which works out to $\int_0^{2\pi}\int_1^\infty \frac{1}{2z^4}dzd\theta$. The integral of this is $\pi/3$.

**0.6.**

- Draw the region for the integral
$$\int_0^1 \int_x^1 e^{y^2}dydx.$$
- Compute the integral (Suggestion: switch order of integration)

**0.7.** *Bonus, Worth no Points* Show that there is no function from the cube to the square

$$f : [0,1] \times [0,1] \times [0,1] \to [0,1] \times [0,1]$$

so that:

- $f(x, y, 0) = (x, y)$
- $f(x, y, t) = (x, y)$ whenever $x, y = 0$ or $1$
- $f(x, y, 1)$ is completely contained in the boundary of the square.