Discussion Notes

In discussion yesterday, I had an incorrect statement about the gradient of the dot product of two vector fields. This doesn't obey a "product rule" that involves curls or divergence of vector fields. Instead, we can associate to a vector field

$$\vec{F}(x,y) = \langle P(x,y), Q(x,y)\rangle$$

a *Jacobian matrix*

$$J_{\vec{F}}(x,y) = \begin{pmatrix} P_x(x,y) & P_y(x,y) \\ Q_x(x,y) & Q_y(x,y) \end{pmatrix}$$

which is taken from thinking of a vector field not as a vector field, but instead as a change of coordinates from (x,y) coordinates to $\langle P, Q\rangle$ coordinates. This is (yet another!) way to take a "derivative" of a vector field.

With this set up, we can show the following "product rule" for gradient:

$$\begin{aligned}
\operatorname{grad}(\vec{F}\cdot\vec{G}) =& \operatorname{grad}(P^F P^G + Q^F Q^G) \\
=& \langle P^F_x P^G + P^F P^G_x + Q^F_x Q^G + Q^F Q^G_x, P^F_y P^G + P^F P^G_y + Q^F_y Q^G + Q^F Q^G_y \rangle \\
=& \begin{pmatrix} P^F_x & Q^F_x \\ P^F_y & Q^F_y \end{pmatrix} \langle P^G, Q^G\rangle + \begin{pmatrix} P^G_x & Q^G_x \\ P^G_y & Q^G_y \end{pmatrix} \langle P^F, Q^F\rangle \\
=& (J_F)^{\perp} G + (J_G)^{\perp} F
\end{aligned}$$

where $M^{\perp}$ is the *transpose* of the matrix (flipping the matrix along its diagonal.)

There are a whole slew of these types of identities for taking gradients, curls and divergence. I wouldn't worry about memorizing any of these identities, but it is possible that you might be asked to derive some of them. For practice, you might try to show that (in dimension 3)

$$\operatorname{div}(\vec{F}\times\vec{G}) = \operatorname{curl}(\vec{F})\cdot\vec{G} - \vec{F}\cdot\operatorname{curl}(\vec{G})$$