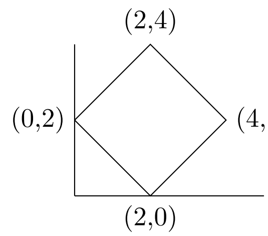
Quiz, March 28th

0.1. Jacobians. Compute the integral of $f(x,y) = xy$ over the following region



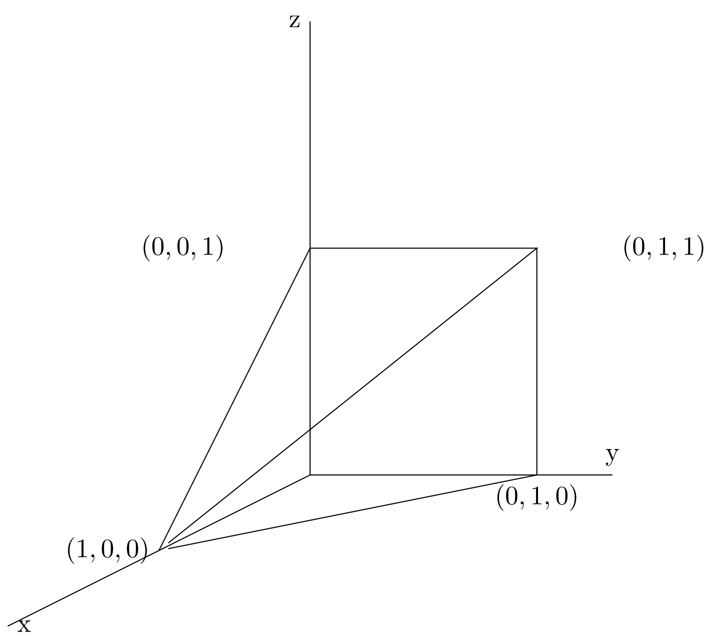
by using the change of coordinates

$$x = u + v \qquad y = u - v$$

Solution: For this, notice that this is given by $u \in [-1,1]$ and $v \in [1,3]$. The jacobian between these two coordinate systems is 2, so our integral converts to

$$\begin{aligned}
\int_{-1}^{1}\int_{1}^{3}(u+v)(u-v)2dvdu =& 2\int_{-1}^{1}\int_{1}^{3}u^2 - v^2dvdu \\
=& 2\int_{-1}^{1} vu^2 - v^3/3|_{v=1}^{v=3}du \\
=& \int_{-1}^{1} 2u^2 - 26/3du \\
=& 2/3u^3 - 26/3u|_{-1}^{1} \\
=& 4/3 - 42/3
\end{aligned}$$

0.2. Setting up triple integrals. Suppose the density of the following solid is given by $\rho(x,y,z) = x + y$.



Set up an integral which computes the mass of the object.

Solution: Let's take the order to be $dzdydx$. Notice that x takes values between 0 and 1, and that $0 \leq z \leq 1 - x$ and $0 \leq y \leq 1 - x$. (In particular, the bounds on z are not dependent on the y coordinate you are at, and the bounds for y are not dependent on the z coordinate. A way to visualize this is that an xz slice of this solid is a square.) Our integral comes out to be

$$\int_0^1 \int_0^{1-x} \int_0^{1-x} x + y dz dy dx$$

0.3. **Cylindrical Coordinates.** Compute the integral of the function $f(x, y, z) = z + x^2 + y^2$ over the region constrained by $0 \leq z \leq 1 - (x^2 + y^2)$. Use cylindrical coordinates.

Solution: This is not only a bound on z by x and y, but also a bound on r in the form of $0 \leq 1 - r^2$. Since r must be positive, we can rewrite this bound as $0 \leq r \leq 1$. Our bounds of integration in cylindrical coordinates are

$$0 \leq r \leq 1$$
$$0 \leq \theta \leq 2\pi$$
$$0 \leq z \leq 1 - r^2$$

Setting up our integral, we have

$$\begin{aligned}
\int_0^{2\pi} \int_0^1 \int_0^{1-r^2} (z + r^2) r dz dr d\theta &= \int_0^{2\pi} \int_0^1 (z^2/2 + r^2 z) r|_0^{1-r^2} dr d\theta \\
&= \int_0^{2\pi} \int_0^1 ((1 - r^2)^2/2 + r^2(1 - r^2)) r dr d\theta \\
&= \int_0^{2\pi} \int_0^1 ((1 - r^2)^2/2 + r^2(1 - r^2)) r dr d\theta \\
&= \int_0^{2\pi} \int_0^1 (1/2 - r^2 + r^4/2 + r^2 - r^4) r dr d\theta \\
&= \int_0^{2\pi} \int_0^1 r/2 - r^5/2 dr d\theta \\
&= 2\pi(1/4 - 1/10)
\end{aligned}$$

Bonus Problem. Worth no credit! A *napkin ring* is made from taking a sphere of radius R, and drilling out of it a cylinder with the same axis of radius r. The resulting napkin ring has a height of $h = \sqrt{R^2 - r^2}$. Show that the volume of a napkin ring only depends on h.