Quiz, March 14th

0.1. **Lagrange Multipliers.** Using the method of Lagrange multiplies, find the area of the largest rectangle that can be drawn with one corner on the origin, and diagonal corner on the constranct $(2x)^2 + (1y)^2 = 1$.
Solution: You want to maximize the function $f(x, y) = xy$ on the constraint $g(x, y) = (2x)^2 + y^2 = 1$. Writing down the Lagrange multiplier equations,

$$\lambda\nabla g = \nabla f$$
$$\langle\lambda\langle 8x, 2y\rangle = \langle y, x\rangle$$

We then have three equations,

$$\lambda 8x = y \qquad \lambda 2y = x \qquad (2x)^2 + y^2 = 1$$

The first two equations give us that $\lambda 2(\lambda 8x) = x$. This tell us that either $16\lambda^2 = 1$, or that $x = 0$.

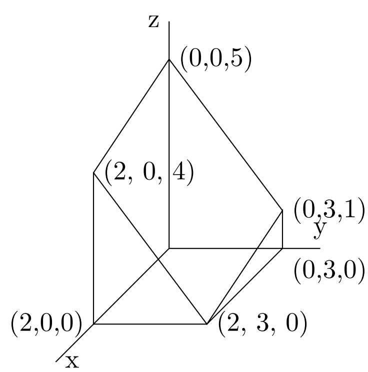
- If $x = 0$, then we also have that $y = 0$, which is not on the ellipse.
- If $\lambda = \pm\frac{1}{4}$, then we have that $y = \pm 2x$, which gives us

  $$(2x)^2 + (\pm 2x)^2 = 1$$

  simplifying to $x = \pm\frac{1}{2\sqrt{2}}$. Then $y = \pm\frac{1}{\sqrt{2}}$.

Both of these results give an area of $\frac{1}{4}$.

0.2. **Computing Double Integrals.** Using double integrals, find the volume of this figure:



Solution: The equation for the plane can be found with the following reasoning. Let $z = ax + by + d$ be the equation of the plane. Then if $x = y = 0$, then $z = 5$. Therefore we know the equation to be

$$z = ax + by + 5$$

If $x = 0, y = 3$ we know that $z = 1$. So $1 = a(0) + b(3) + 5$, so $b = -4/3$. If $y = 0$ and $x = 2$, then $z = 4$, so $4 = a(2) + 0 + 5$, so $a = -1/2$. Therefore the equation of the plane is

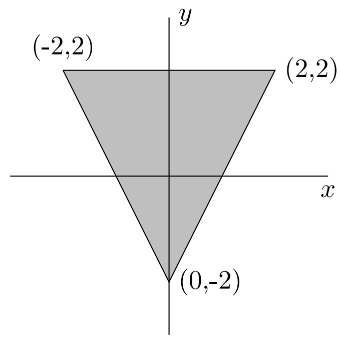
$$z = -x/2 - 4y/3 + 5$$

We integrate this over the bounds written

$$\int_{x=0}^{2}\int_{y=0}^{3} -x/2 - 4y/3 + 5 dydx$$

which comes out to 15.

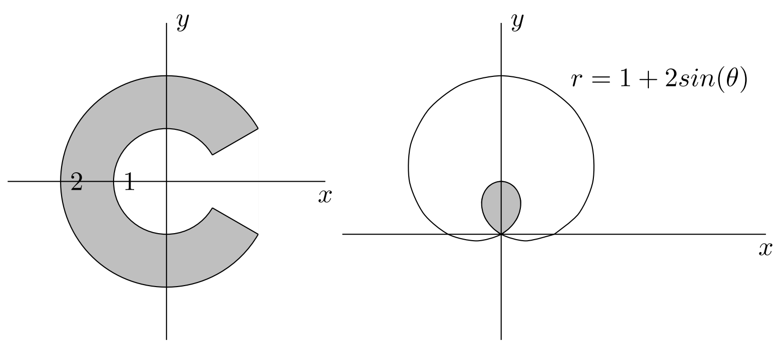
0.3. **Setting up Double Integrals.** Set up an integral that will compute the volume of a function $f(x, y)$ over each of the following regions:



Solution: We can do this as a single integral integrating with respect to x first, then y. We have that the left bound for x is given by $x_l = -y/2 - 1$, and the right bound is given by $x_r = y/2 + 1$. So , the bounds are

$$\int_{y=-2}^{2} \int_{x=-y/2-1}^{x=y/2+1} f(x, y) dx dy$$

Set up an integral that will compute the volume of a function $g(r : \theta)$ over each of the following regions:



The first one is given by $\int_{\theta=\pi/6}^{\theta=11\pi/6} \int_{r=1}^{2} f(r : \theta) r dr d\theta$. The second one is more tricky. Here is the long explanation: How do we integrate over this region? One of the subtle difficulties of integrating over polar region is that

Let's look at the boundary of the region given by this example. Thee inner loop of the function $1+2\sin(\theta)$ is drawn out when θ goes from $7\pi/6$ to $11\pi/6$. However, on this region the function $2\sin(\theta) + 1$ is *negative.* Because the bounds of integration for a polar function should only include positive r values, it would be *incorrect* to set up this integral as

$$\int_{\theta=7\pi/7}^{\theta=11\pi/6} \int_{0}^{r=1+2\sin(\theta)} f(r, \theta) \;\; r dr d\theta /$$

To correctly write down this polar integral, we need to express the boundary for r by a polar function $r(\theta)$ which is always greater than or equal to zero. One can check that

$$2\sin(\theta) - 1$$

also graphs out the same curve. However, for the function $2\sin(\theta) - 1$, the inner region is graphed when θ goes from $\pi/6$ to $5\pi/6$, and r is graphed positively over this region. So the correct way to integrate over this region is by the polar integral:

$$\int_{\theta=\pi/6}^{\theta=5\pi/6} \int_{0}^{r=2\sin(\theta)-1} f(r, \theta) \;\; r dr d\theta$$

Bonus Problem. *Worth no points!* Can you find a set of numbers a_{ij} where $i, j \in \mathbb{N}$, so that

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij} = 1$$

but

$$\sum_{j=1}^{\infty} \sum_{i=1}^{\infty} a_{ij} = 0.$$

Use these functions to describe a function $f(x, y)$ so that $\iint f dxdy \neq \iint f dydx$. (This will be in an improper integral.)