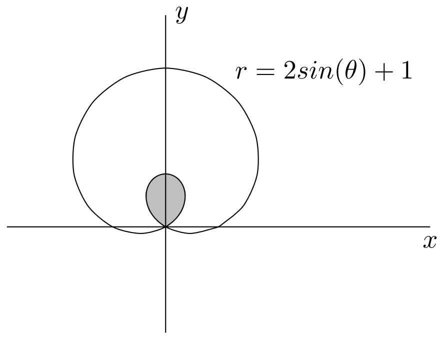
NOTES, MARCH 9TH



How do we integrate over this region? One of the subtle difficulties of integrating over polar region is that

- Functions that take values in polar coordinates (which describe the boundaries of the regions) are allowed to take on negative values for r. In this example $r(\theta) = 2\sin(\theta)+1$ is a perfectly good way of describing the boundary of the region we want to integrate over, even though it takes on a negative value for some values of θ.
- Functions $f(r,\theta)$, with polar inputs, are *technically* only well defined when $r \geq 0$. [1]

Let's look at the boundary of the region given by this example. Thee inner loop of the function $1 + 2\sin(\theta)$ is drawn out when θ goes from $7\pi/6$ to $11\pi/6$. However, on this region the function $2\sin(\theta) + 1$ is *negative.* Because the bounds of integration for a polar function should only include positive r values, it would be *incorrect* to set up this integral as

$$\int_{\theta=7\pi/7}^{\theta=11\pi/6} \int_{0}^{r=1+2\sin(\theta)} f(r,\theta) \;\; r dr d\theta /$$

To correctly write down this polar integral, we need to express the boundary for r by a polar function $r(\theta)$ which is always greater than or equal to zero. One can check that

$$2\sin(\theta) - 1$$

also graphs out the same curve. However, for the function $2\sin(\theta) - 1$, the inner region is graphed when θ goes from $\pi/6$ to $5\pi/6$, and r is graphed positively over this region. So the correct way to integrate over this region is by the polar integral:

$$\int_{\theta=\pi/6}^{\theta=5\pi/6} \int_{0}^{r=2\sin(\theta)-1} f(r,\theta) \;\; r dr d\theta$$

How to generally do this? In general, let $r(\theta)$ be a bound for integration, and let's suppose that it is negative on $\theta \in [a, b]$. Then we want to replace it with a function which draws out the same boundary, but is instead defined over the region $\theta \in [-b, -a]$ and takes on positive values. A new candidate function is

$$r_{new}(\theta + \pi) = -r(\theta)$$

For instance, if $r = 1 + 2\sin(\theta)$, one can check that

$$-r(\theta) = -(1 + 2\sin(\theta)) = -1 - 2\sin(\theta) = 2\sin(\theta + \pi) - 1 = r_{new}(\theta + \pi)$$

where $r_{new}(\theta) = 2\sin(\theta) - 1$.

Concept Check. See if you can set up the integral for the area between the two loops.

[1] You can remove this technical assumption if it is that case that $f(r,\theta) = f(-r,-\theta)$, but this is not going to necessarily be the case with some function.