Lagrange Multipliers

0.1. **Lagrange Multipliers.** Find the points on the curve $xy = 1$ where the function $x + y$ is (locally) maximized and minimized.

Solution: Lagrange's equations ask us to solve where $f(x, y) = x + y$ has gradient in the same direction as $g(x, y) = xy = 1$. We compute

$$\begin{aligned}\nabla g =&\langle y, x\rangle \\ \nabla f =&\langle 1, 1\rangle\end{aligned}$$

Then Lagrange's equations give us the following three equalities:

$$\begin{aligned}\lambda y =&1 \\ \lambda x =&1 \\ xy =&1\end{aligned}$$

From the first two equations we get that $1/\lambda = x = y$, which plugs into the third equation as

$$x \cdot x = 1$$

Which tells us that $x = \pm 1$. Using this solution for x to get y and λ again, we have 2 solutions, where

$$x = 1, y = 1, \lambda = 1$$

or

$$x = -1, y = -1, \lambda = -1$$

Checking the equation show thathere is a maximum at $1, 1$, where $f(1, 1,) = 2$, and a minimum where $f(-1, -1) = -2$.

0.2. **Lagrange Multipliers.** Show that the point on the unit sphere which is closest to the point (a, b, c) is

$$\left(\frac{a}{\sqrt{a^2 + b^2 + c^2}}, \frac{b}{\sqrt{a^2 + b^2 + c^2}}, \frac{c}{\sqrt{a^2 + b^2 + c^2}}\right)$$

Solution: Minimizing the distance is the same a minimizing the distance squared. So, we are trying to minimuze the function

$$f(x, y, z) = (x - a)^2 + (y - b)^2 + (z - c)^2$$

on the constraint

$$g(x, y, z) = x^2 + y^2 + z^2 = 1$$

We compute

$$\begin{aligned}\nabla g =&\langle 2x, 2y, 2z\rangle \\ \nabla f =&\langle 2(x - a), 2(y - b), 2(z - c)\rangle\end{aligned}$$

Then Lagrange's equations give us the following three equalities:

$$\begin{aligned}\lambda 2x =&2(x - a) \\ \lambda 2y =&2(y - b) \\ \lambda 2z =&2(z - c)\end{aligned}$$

$$x^2 + y^2 + z^2 = 1$$

This shows $x = a/(1 - \lambda)$, $y = b/(1 - \lambda)$, $z = c/(1 - \lambda)$, so by the third equation

$$a^2 + b^2 + c^2 = (1 - \lambda)^2$$

giving us

$$\lambda = 1 \pm \sqrt{a^2 + b^2 + c^2}$$

telling us that (x, y, z) is

$$\pm\left(\frac{a}{\sqrt{a^2+b^2+c^2}}, \frac{b}{\sqrt{a^2+b^2+c^2}}, \frac{c}{\sqrt{a^2+b^2+c^2}}\right)$$

0.3. **Lagrange Multipliers.** Use the method of Lagrange Multipliers to find the point on the sphere which is closest to the plane

$$ax + by + cz + d = 0.$$

(Be sure to check where the gradient is zero, what does this mean?) **Solution:**Minimizing the distance is the same a minimizing the distance squared. So, we are trying to minimuze the function

$$f(x, y, z) = \frac{(ax + by + cz + d)^2}{a^2 + b^2 + c^2}$$

on the constraint

$$g(x, y, z) = x^2 + y^2 + z^2 = 1$$

We compute

$$\begin{aligned} \nabla g =& \langle 2x, 2y, 2z\rangle \\ \nabla f =& 2(ax + bx + cz + d)\langle 2a, 2b, 2c\rangle \end{aligned}$$

Then Lagrange's equations give us the following three equalities:

$$\begin{aligned} \lambda 2x =& 2(ax + bx + cz + d)a \\ \lambda 2y =& 2(ax + bx + cz + d)b \\ \lambda 2z =& 2(ax + bx + cz + d)b \end{aligned}$$

$$x^2 + y^2 + z^2 = 1$$

One can check that either $ax + bx + cz + d = 0$ and $x^2 + y^2 + z^2 = 1$, which means that there are points in the intersection of the plane an the sphere, or (x, y, z) is

$$\pm\left(\frac{a}{\sqrt{a^2+b^2+c^2}}, \frac{b}{\sqrt{a^2+b^2+c^2}}, \frac{c}{\sqrt{a^2+b^2+c^2}}\right)$$