PRACTICE MIDTERM, FEB 29

NAME:

0.1. The line AB is given by $\vec{a}(t) = \langle 2t+1, 3t+1, 2\rangle$ and the line CD is given by $\vec{b}(s) = \langle s, s, s\rangle$.

(a) Find a line ℓ which intersects both the line AB and the line CD at right angles.
Solution: We can find the direction of this line by taking the cross productfo the directions that the other two lines travel in. The first line travels in the direction

$$\langle 2, 3, 2\rangle$$

and the second line travels in the direction

$$\langle 1, 1, 1\rangle$$

This means that the direction that I line which is perpendicular to both of these must travel in the direction

$$\langle 2, 3, 0\rangle \times \langle 1, 1, 1\rangle$$

This cross product is $\langle 3, -2, -1\rangle$.
Then to find the line which intersects both of these, and travels in the same direction, we need check wher e

$$\langle 2t+1, 3t+1, 2\rangle + q\langle 3, -2, -1\rangle = \rangle s, s, s\langle$$

for some t, q, s. Writing these as a system of equations

$$\begin{aligned} 2t + 3q - s + 1 =& 0 \\ 3t - 2q - s + 1 =& 0 \\ 0t - q - s + 2 =& 0 \end{aligned}$$

Solving this system of equations gives $t = 5/14$, which tells us an initial point. Our equation for a line is

$$\ell(t) = \vec{a}(5/14) + q\vec{v}$$

or

$$\langle 24/14 + 3q, 29/14 - 2q, 2 - q.$$

(b) Find a function $D(s,t)$ which computes the distance between $\vec{a}(t)$ and $\vec{b}(s)$. Minimize this function to find the smallest distance between the lines AB and CD.
Solution: The function that computes the distance between these two curves is

$$\sqrt{(2t+1-s)^2 + (3t+1-s)^2 + (2-s)^2}$$

And minimizing this function is the same as minimizing the function squared. So we are looking to minimize

$$f(t,s)(2t+1-s)^2 + (3t+1-s)^2 + (2-s)^2$$

Taking the derivative with respect to t, we get

$$f_t(t,s) = 4(2t+1-s) + 6(3t+1-s) = 10 - 10s + 26t$$

and takind the derivative with respect to s we get

$$f_s(t,s) = -2(2t+1-s) - 2(3t+1-s) - 2(2-s) = -8 + 6s - 10t$$

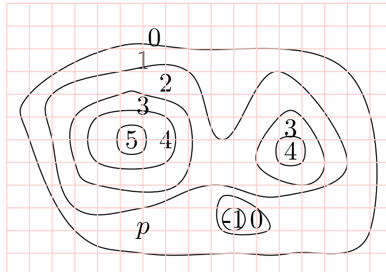
Setting both of these equal to zero, and solving for s and t we get

$$t = 5/14, s = 27/14.$$

which is the same intercept as in the previous problem.

(c) Confirm that the line ℓ minimizes the distance between AB and CD.

0.2. Draw a contour plot for a function which has exactly 4 critical points– 1 saddle, 2 maximums and 1 minimum. **Solution:**



0.3. The upper hemisphere is given by the function $f(x, y) = \sqrt{1 - x^2 - y^2}$. Using the gradient, compute the tangent plane to this graph at the point $(a, b, \sqrt{1 - a^2 - b^2})$.
Solution: We compute the gradient of $f(x, y)$ to be

$$\nabla f(x, y) := \left\langle \frac{-x}{\sqrt{1 - x^2 - y^2}}, \frac{-y}{\sqrt{1 - x^2 - y^2}} \right\rangle$$

The equation for the tangent plane using the gradient is

$$(z - z_0) = \nabla f \cdot \langle x - x_0, y - y_0 \rangle$$

so, we get that the equation for the tangent plane is

$$z - \sqrt{1 - a^2 - b^2} = (x - a)\frac{-a}{\sqrt{1 - a^2 - b^2}} + (y - b)\frac{-b}{\sqrt{1 - a^2 - b^2}}$$

If you want to simplify by multiplyfing by the root, we get

$$z\sqrt{1 - a^2 - b^2} - 1 = -a(x - a) - b(x - b)$$

Simplifying further gives

$$a(x - a) + b(x - b) + z\sqrt{1 - a^2 - b^2} = 1$$

Notice that the normal vector to the tangent plane at a point on the sphere is $\langle a, b, \sqrt{1 - a^2 - b^2} \rangle$, which *is the point which the plane is tangent to.*

0.4. Let $f(x, y) = x^2 + y^2$. Suppose that we know that $\vec{r}(t) = \langle x(t), y(t) \rangle$ has

$$\begin{aligned} |\vec{r}(0)| &= 0 \\ |\vec{r'}(0)| &= 1 \\ |\vec{r''}(0)| &= 0 \end{aligned}$$

Compute

$$\frac{d^2}{dt^2} f(x(t), y(t)).$$

$$\frac{d^2}{dt^2} f(x(t), y(t)) = \frac{d}{dt}(f_x|_{(x(t),y(t))} x'(t) + f_y|_{(x(t),y(t))} y'(t))$$

You have two choices here. We could either continue computing the chain rule abstractly, or substitute what we know for f_x and f_y. Let's first try substitution. Notice that $f_x = 2x$ and $f_y = 2y$.

$$= \frac{d}{dt}(2x(t)x'(t) + 2y(t)y'(t))$$

Using the Product Rule

$$= 2x'(t)x'(t) + 2x(t)x''(t) + 2y'(t)y'(t) + 2y(t)y''(t)$$

We know from assumption that $x''(0) = y''(0) = 0$.

$$
\begin{aligned}
&= 2x'(t)x'(t) + 2y'(t)y'(t) \\
&= 2\langle x'(t), y'(t)\rangle \cdot \langle x'(t), y'(t)\rangle \\
&= 2|r'(t)|^2 \\
&= 2(1)^2 = 2
\end{aligned}
$$

The other option would have been to continue using the chain rule. You would have gotten from the product rule:

$$= \frac{df_x}{dt}x'(t) + f_x x''(t) + \frac{df_y}{dt}y'(t) + f_y y''(t)$$

Notice that $y''(0) = x''(0) = 0$.

$$= \frac{df_x}{dt}x'(t) + \frac{df_y}{dt}y'(t)$$

Applying the Chain Rule Again;

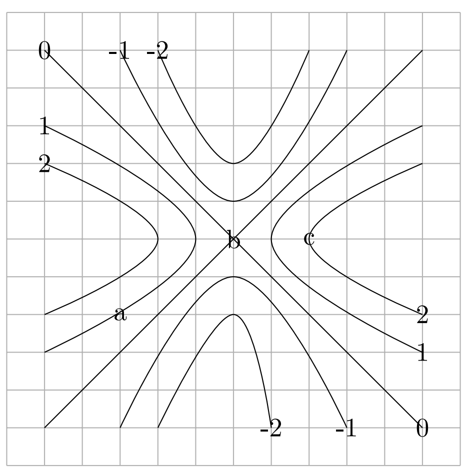
$$= f_{xx}x'(t)x'(t) + 2f_{xy}x'(t)y'(t) + f_{xy}x'(t)y'(t) + fyyy'(t)y'(t)$$

We know that $f_{xx}(0,0) = f_{yy}(0,0) = 2$ and $f_{xy}(0,0) = 0$

$$
\begin{aligned}
&= 2\langle x'(t), y'(t)\rangle \cdot \langle x'(t), y'(t)\rangle \\
&= 2|r'(t)|^2 \\
&= 2(1)^2 = 2
\end{aligned}
$$

which should always be 1.

0.5. Estimate the Gradient at each of the points. Each grid length is 1 unit.



Solution:

(a) This one is the trickiest. It looks like ∇f is in the direction of $\langle -3, 2\rangle$, based on the fact that the gradient must be perpendicular to the normal curve. The length of ∇f is given by the

$$|\nabla f| = \frac{1}{\text{The directional derivative of } f \text{ in the direction of } \nabla f.}$$

Condsider a vector starting at a, then travelling perpendicularly to the next level curve. This vector has a length (or a "run") of about 1, and a "rise" of about 1. Therefore, we estimate

$$D_{\nabla f/|\nabla f|}(a) \sim 1$$

which means that $|\nabla f| \sim 1$. We then have that $\nabla f \sim \langle -3/\sqrt{13}, 2/\sqrt{13} \rangle$

(b) This is a critical point, so the gradient is $\langle 0, 0 \rangle$.

(c) This will have gradient only in the x direction, $\langle 1, 0 \rangle$.

0.6. A. n ant travels in $x - y$ coordinates along the path $(3t, t^2)$ from time 0 to 2. It walks along the hill $f(x, y) = 2 - x^2 - y$ during this time.

- How long is the path that the ant travels along the hill, and what is its maximal speed?
- When (if ever) does the ant travel perpendicular to the gradient of the hill?

Solution: The ant travels along the path (with the hill)

$$\langle 3t, t^2, 2 - (3t)^2 - t^2 \rangle = \langle 3t, t^2, 2(1 - 5t^2) \rangle$$

The velocity of the ant is $\vec{v}(t) = \langle 3, 2t, 20t \rangle$ The length of this vector is $|\vec{v}(t) = \sqrt{3^2 + 404t^2}$ and the length of the curve is

$$\int_0^2 \sqrt{3^2 + 404t^2} dt$$

The maximal velocity is when this function has a maximum on the interval, which is at time 2.
For the second part, notice that

$$\frac{d}{dt}(f(x(t), y(t)) = f_x x' + f_y y' = \nabla f \cdot \langle x', y' \rangle$$

so we are checking when $\langle -2x(t), -1 \rangle \cdot \langle 3, 2t \rangle = \langle -6t, -1 \rangle \cdot \langle 3, 2t \rangle = 0$.
This occurs when $t = 0$.