Discussion Problems, Max and Min

0.1. **Max, Min Calculation.** The Eggshell Bedding is given by the function

$$f(x, y) = \sin(x)\sin(y)$$

Find the local maximas, minimas and saddle points of this function. Use this data to create a "rough" contour plot of this function.

0.2. **Setting up a Max-Min problem.** The function $f(x, y) = \sqrt{1 - x^2 - y^2}$ graphs an upper hemisphere centered at the origin. Find the point on the upper hemisphere which is the closest to the plane

$$z = 2x + 2y + 5.$$

0.3. **Max, Min and Chain Rule.** Suppose that $f(x, y)$ has a maximum at the origin which can be detected by the second derivative test. Show that for every line through the origin $x(t) = at$, and $y(t) = bt$, we have

$$\frac{d^2}{dt^2} f(x(t), y(t)) < 0$$

0.4. **Max, Min Calculation.** The Eggshell Bedding is given by the function

$$f(x, y) = \sin(x)\sin(y)$$

Find the local maximas, minimas and saddle points of this function. Use this data to create "rough" contour plot of this function.
Solution: To find the local maximas, minimas and saddle points, we first need to find all of the critical values. This means looking for where the gradient is 0.

$$\nabla f(x, y) = \langle \cos(x)\sin(y), \sin(x)\cos(y) \langle$$

Let's also compute some second derivatives to determine if these critical points are max/min/saddle:

$$f_{xx} = -\sin(x)\sin(y) \qquad f_{xy} = \cos(x)\cos(y) \qquad f_{yy} = -\sin(x)\sin(y)$$

This gradient is zero whenever one of the following conditions are met :

- We have $\sin(y) = \sin(x) = 0$.

  $$x = k\pi \quad y = l\pi$$

  Applygin the second derivative test

  $$\begin{aligned} D =& f_{xx}f_{yy} - (f_{xy})^2 \\ =& (\sin(k\pi)\sin(k\pi))(\sin(k\pi)\sin(k\pi)) - (cos(k\pi)\cos(k\pi))^2 \\ =& -1 \end{aligned}$$

  which means that these are saddles.
- We have $\cos(x) = \cos(y) = 0$ This happens when

  $$x = \pi/2 + k\pi \quad y = \pi/2 + l\pi$$

  Now, we try to compute using the second derivative test

  $$\begin{aligned} D =& f_{xx}f_{yy} - (f_{xy})^2 \\ =& (\sin(\pi/2 + k\pi)\sin(\pi/2 + k\pi))(\sin(\pi/2 + k\pi)\sin(\pi/2 + k\pi)) - (cos(\pi/2 + k\pi)\cos(\pi/2 + k\pi))^2 \\ =& 1 \end{aligned}$$

  So, we either have a max or a min. We have a max when $k + l$ is even, and a minimum when $k + l$ is odd.

0.5. **Max, Min and Chain Rule.** Suppose that $f(x, y)$ has a maximum at the origin detectable by second derivative test. Show that for every path $x(t) = at$, and $y(t) = bt$, we have

$$\frac{d^2}{dt^2} f(x(t), y(t)) < 0$$

Solution: We have

$$\begin{aligned} \frac{d^2}{dt^2} f(x(t), y(t)) =& \frac{d}{dt}\left(\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}\right) \\ =& \left(\frac{d}{dt}\frac{\partial f}{\partial x}\right)\frac{dx}{dt} + \frac{\partial f}{\partial x}\left(\frac{d}{dt}\frac{dx}{dt}\right) + \left(\frac{d}{dt}\frac{\partial f}{\partial y}\right)\frac{dy}{dt} + \frac{\partial f}{\partial y}\left(\frac{d}{dt}\frac{dy}{dt}\right) \end{aligned}$$

Let's make some substitutions. The second derivatives of x and y are 0, and we know the first derivatives of x and y to be a and b

$$= \left(\frac{d}{dt}\frac{\partial f}{\partial x}\right) a + \left(\frac{d}{dt}\frac{\partial f}{\partial y}\right) b$$

Applying the Chain Rule Again

$$= \left(\left(\frac{\partial}{\partial x}\frac{\partial f}{\partial x}\right)\frac{dx}{dt} + \left(\frac{\partial}{\partial y}\frac{\partial f}{\partial x}\right)\frac{dy}{dt}\right) a + \left(\left(\frac{\partial}{\partial x}\frac{\partial f}{\partial y}\right)\frac{dx}{dt} + \left(\frac{\partial}{\partial y}\frac{\partial f}{\partial y}\right)\frac{dy}{dt}\right) b$$

Combining Mixed Partials, and substituting in values for x' and y'

$$\begin{aligned} =& \left(\frac{\partial^2 f}{\partial x \partial x} a + \frac{\partial^2 f}{\partial x \partial y} b\right) a + \left(\frac{\partial^2 f}{\partial y \partial x} a + \frac{\partial^2 f}{\partial y \partial y} b\right) b \\ =& a^2 \frac{\partial^2 f}{\partial x \partial x} + 2ab\frac{\partial^2 f}{\partial x \partial y} + b^2 \frac{\partial^2 f}{\partial y \partial y} \end{aligned}$$

For this to be strictly negative, we need this to have no zeros; this means that the discriminant of this polynomial (in variables a, b) needs to be negative. This means that

$$\left(2\frac{\partial^2 f}{\partial x \partial y}\right)^2 - 4\frac{\partial^2 f}{\partial x \partial x}\frac{\partial^2 f}{\partial y \partial y} \leq 0$$

which we know because the second derivative test finds this.

0.6. **Setting up a Max-Min problem.** The function $f(x, y) = \sqrt{1 - x^2 - y^2}$ graphs an upper hemisphere centered at the origin. First, set up an equation whose minimization in two variables will give us the closest point. Then find the point on the upper hemisphere which is the closest to the plane

$$z = 2x + 2y + 5.$$

Solution: We use the formula for the distance of a point from a plane:

$$d_p(x, y) = \frac{ax_0 + by_0 + cz_0 + d}{\sqrt{a^2 + b^2 + c^2}}$$

Here, a, b, c, d are constants definying the plane, so that $a = 2, b = 2, c = -1, d = 5$. The points x_0, y_0 and z_0 will be on the sphere.

$$d_p(x, y) = \frac{(2x + 2y - \sqrt{1 - x^2 - y^2} + 5)}{\sqrt{2^2 + 2^2 + 1^2}} = \frac{(2x + 2y - \sqrt{1 - x^2 - y^2} + 5)}{\sqrt{9}}$$

It is equivalent to finding the minimum of this function:

$$\bar{d} := (2x + 2y - \sqrt{1 - x^2 - y^2} + 5)$$

Taking partial derivatives we get

$$\frac{\partial \bar{d}}{\partial x} = 2 - \frac{x}{\sqrt{1 - x^2 - y^2}}$$

and

$$\frac{\partial \bar{d}}{\partial y} = 2 - \frac{y}{\sqrt{1 - x^2 - y^2}}$$

These are both suppose to be equal to 0. We have

$$\begin{aligned} x/2 &= \sqrt{1 - x^2 - y^2} \\ y/2 &= \sqrt{1 - x^2 - y^2} \end{aligned}$$

Which gives us $x = y$. Making the substitution

$$\begin{aligned} 2 =& \frac{y}{\sqrt{1 - y^2 - y^2}} \\ =& \frac{y}{\sqrt{1 - 2y^2}} \\ 4(1 - 2y^2) =& y^2 \\ 9y^2 =& 4 \end{aligned}$$

so that $y = 2/3$. Likewise, $x = 2/3=$