Discussion Notes

We've been messing around with the definition of differentiable for a while now, because it's a pretty hard concept to define. It'll take some set up to make a definition.

Let $p = (x_0, y_0)$ be a point, $f(x, y)$ be a function, and $\vec{v} = \langle v_x, v_y \rangle$ be a vector. Define the derivative of f by $\vec{v}$ at p to be

$$(D_p f)(\vec{v}) := \frac{d}{dt} f(p + t\vec{v})$$

where $p + t\vec{v} = (x_0 + v_x t, y_0 + v_y t)$. Notice that if $\vec{v}$ is a unit vector, then this is the directional derivative of f in the direction of v. Otherwise, this is the derivative of f as applied to the path $p + t\vec{v}$.
Notice that $(D_p f)$ is a *function of* $\vec{v}$. (Recall, a function is nothing more than a rule that prescribes to every input an output; in this case, the input if $\vec{v}$ and the output is $\frac{d}{dt} f(p + t\vec{v})$. We could write this as

$$D_p f : \mathbb{R}^2 \to \mathbb{R}$$

where $\mathbb{R}^2$ is the vector space which contains $\vec{v}$.

Definition: If $D_p f : \mathbb{R}^2 \to \mathbb{R}$ is a *linear map*, then f is called a *differentiable* function.

One easy way to see that differentiability defines this condition is that we can define this as

$$\begin{aligned} D_p f : \mathbb{R}^2 &\to \mathbb{R} \\ \vec{v} &\mapsto \nabla f \cdot \vec{v} \end{aligned}$$

and the dot product is a linear map.