Discussion Problems, Gradients and Directional Derivatives

0.1. **Gradient Calculation.** Jack and Jill walk up the hill $f(x,y)$, to fetch a pail of water. The path Jack takes is

$$r(t) = (3t, 2t)$$

and the path that Jill takes is

$$s(t) = (-2t, 3t)$$

Jack reports that at time 0,

$$\frac{d}{dt}(f(3t,2t))\bigg|_{t=0} = 1$$

and Jill reports that

$$\frac{d}{dt}(f(-2t,3t))\bigg|_{t=0} = 2$$

- What is the gradient $\nabla f(0,0)$.
- Suppose additionally that $f(0,0) = 2$. What is the tangent plane to the graph of f at $(0,0,2)$?

Solution: From the chain rule (or definition of directional derivative) we have

$$\begin{aligned} 1 =& \frac{d}{dt}\frac{d}{dt}(f(3t,2t))\bigg|_{t=0} \\ =& \frac{\partial f}{\partial x}\bigg|_{(0,0)} \frac{d(3t)}{dt}\bigg|_{t=0} + \frac{\partial f}{\partial y}\bigg|_{(0,0)} \frac{d(2t)}{dt}\big|_{t=0} \\ =& 3\,\frac{\partial f}{\partial x}\bigg|_{(0,0)} + 2\frac{\partial f}{\partial y}\bigg|_{(0,0)} \end{aligned}$$

Similarly, Jill's path tells us that

$$2 = -2\,\frac{\partial f}{\partial x}\bigg|_{(0,0)} + 3\,\frac{\partial f}{\partial y}\bigg|_{(0,0)}.$$

This gives us a system of equations for the partial derivatives at $(0,0)$.

$$\begin{aligned} 2 =& -2\,\frac{\partial f}{\partial x}\bigg|_{(0,0)} + 3\,\frac{\partial f}{\partial y}\bigg|_{(0,0)} \\ 1 =& 3\,\frac{\partial f}{\partial x}\bigg|_{(0,0)} + 2\frac{\partial f}{\partial y}\bigg|_{(0,0)} \end{aligned}$$

Solvig this system of linear equations gives that $\frac{\partial f}{\partial x}\Big|_{(0,0)} = -1/13$ and $\frac{\partial f}{\partial y}\Big|_{(0,0)} = 8/13$. From here, we know that the gradient it

$$\nabla f(0,0) = \langle -1/13, 8/13\rangle$$

and the formula for the tangent plane, $(z - z_0) = \frac{\partial f}{\partial x}(x - x_0) + \frac{\partial f}{\partial y}(y - y_0)$ gives that the tangent plane is

$$z = -x/13 + 8y/13 + 2$$

0.2. **Directional Derivatives of the Monkey Saddle.** The pointy Monkey saddle is given by the function

$$f(x,y) = \frac{y^3 - 3x^2 y}{x^2 + y^2}$$

In polar coordinates, this is given by $r\cos(3\theta)$.

- Let $\vec{v} = \langle \cos\theta, \sin\theta\rangle$. Compute the directional derivative

  $$D_{\vec{v}} f|_{(0,0)}.$$

  Hint: What should the directional derivatives at the origin of a function in polar form be?
- Using the above, compute the *Gradient* of $f(x,y)$ at the origin.
- Show that it is *not* the case that $D_{\vec{v}} = \nabla f \cdot \vec{v}$.
- What went wrong?

Solution: Using the hint, notice that the directional derivative in the direction of $\vec{v} = \langle \cos\theta, \sin\theta \rangle$ of a function $f(r, \theta)$ at the origin (and it is important here that we are at the origin!) is simply given by $\frac{\partial f}{\partial r}|_{(r=0,\theta)}$ This means that

$$D_{\vec{v}} f = \cos(3\theta)$$

Now, the *gradient* at the origin is given by $\langle \partial_x f, \partial_y f$. The derivative in the x direction can be found by

$$\frac{d}{dt} f(t, 0)\bigg|_{t=0} = \frac{d}{dt}(0) = 0$$

Similarly, $\partial_y f(0, 0) = 1$.
So the Gradient is $\langle 0, 1 \rangle$.