Quiz, Feb 22

Name:

0.1. **Continuity.** Prove or disprove the continuity of the following function:

$$f(x,y) = \frac{x^2 + 2xy + y^2}{x^2 + y^2}$$

(That is, prove or disprove the existence of a limiting value at $(0,0)$) **Solution:**It is not continuous; we can take two limits and show that they do not agree.
If we take the limit along the line $(t,0)$, we get that

$$\lim_{t \to 0} f(t,0) = t^2/t^2 = 1$$

If instead we take the limit along the line (t,t), we get that

$$\lim_{t \to 0} f(t,t) = 3t^2/2t^2 = 3/2$$

so the limit cannot exist.

0.2. **Partial Derivatives, I.** Using partial derivatives at $(0,0)$, estimate the value of

$$f(x,y) = \cos(x)\sin(y)$$

at the point $f(1/2,1)$. **Solution:**The derivative in the x direction is

$$\partial_x f = -\sin(x)\sin(y)$$

The derivative in the y direction is

$$\partial_y f = \cos(x)\cos(y)$$

So, we have that $\partial_x f(0,0) = 0$ and $\partial_y f(0,0) = 1$. So the estimate (by linear approximation) is

$$f(1/2,1) \simeq f(0,0) + 1/2\partial_x f(0,0) + 1\partial y f(0,0) = 0 + 0 + 1 = 1.$$

0.3. **Chain Rule.** Use the chain rule, and the function $f(x, y) = x/y$ to show the quotient rule:

$$\frac{d}{dt}\left(\frac{x(t)}{y(t)}\right) = \frac{y(t)x'(t) - x(t)y'(t)}{(y(t))^2}$$

Your proof may *not* use the chain rule. You may use the power rule in your proof.
Solution: If we take the derivative of

$$\begin{aligned}
\frac{d}{dt}\left(\frac{x(t)}{y(t)}\right) =& \frac{d}{dt} f(x(t), y(t)) \\
=& \left.\frac{\partial f}{\partial x}\right|_{(x(t),y(t))} \left.\frac{dx(t)}{dt}\right|_t + \left.\frac{\partial f}{\partial y}\right|_{(x(t),y(t))} \left.\frac{dy(t)}{dt}\right|_t \\
=& \left(\frac{1}{y(t)}\right)(x'(t)) + \left(\frac{-x(t)}{(y(t))^2}\right) y'(t) \\
=& \frac{y(t)x'(t) - x(t)y'(t)}{(y(t))^2}
\end{aligned}$$

Bonus Problem. *Worth no points!* Can you find $f(x, y)$, a function of two variables , so that

- $f(0, 0) = 0$
- $\lim_{t\to 0} f(at, bt) = 0$ for all a, b. (This means the limit value of $f(x, y)$ along every line approaching the origin is 0.)
- f is *not* continuous at $(0, 0)$!

Solution: Consider the function which is 1 where $y = x^2$, except at the origin, and 0 everywhere else. Every line through the origin intersects the parabola in at most 2 points: once at the origin, and perhaps at one other location. This means that for any line through the origin, there is a value t_{small} so that

$$f(at, bt) = 0 \quad \text{for all } t \text{ smaller than } t_{small}.$$

This means that all lines eventually see this function as zero, so the limit along any line approaching the origin is zero.
However, the limit approaching the origin along the curve (t, t^2) see a limiting value of 1. So, the function is not continious.