1. Derivatives of Multi-variable functions, I

1.1. **Multivariable Chain Rule I.** Suppose that $f(x, y)$ is a differentiable function of two variables, and we know its differential at a point to be:

$$(\mathrm{d}f)(1,1) = 2\mathrm{d}x + 3\mathrm{d}y.$$

Compute $\frac{\mathrm{d}}{\mathrm{d}t} f(t+1, t^2+1)\big|_{t=0}$.
Solution: From the differential, we know that

$$\partial_x f(1,1) = 2 \quad \partial_y f(1,1) = 3$$

and from the chain rule, we know that

$$\frac{d}{dt} f(x(t), y(t)) = \partial_x f x'(t) + \partial_y f y'(t).$$

Because $(x(0), y(0)) = (1,1)$, we may just substitute in the values of the partial derivatives that we know. Furthermore

$$x'(0) = 1 \quad y'(0) = 0$$

So,

$$\frac{d}{dt} f(x(t), y(t))|_{t=0} = 2(1) + 3(0) = 2.$$

1.2. **Multivariable Chain Rule II.** The product rule in single variable calculus states that $(fg)' = f'g + g'f$. Derive this using the chain rule from multivariable calculus. (This is an important exercise! Make sure you are able to identify at every step exactly what each derivative you are writing down means.)
Solution: The idea is to consider the function

$$M(f, g)) = fg$$

Then

$$\begin{aligned}\frac{d}{dt}(f(t)g(t)) =& \frac{d}{dt} M(f(t), g(t)) \\ =& \partial_f M f'(t) + \partial_g M g'(t)\end{aligned}$$

Notice $\partial_f M = \partial_f (fg) = g$

$$=g(t)f'(t) + f(t)g'(t)$$

1.3. **Differential I.** Is it possible for the differential of a function to be

$$\mathrm{d}f = (2x+3)\mathrm{d}x + (2x+3)\mathrm{d}y$$

If so, find a function which has this differential. If not, why?

Solution: If f_x and f_y are continuous derivatives of a function f, then $f_{xy} = f_{yx}$ (clairaut's theorem). But $f_{xy} = 0$ and $f_{yx} = 2$, so these cannot be the derivatives of a function.