1. Derivatives of Multi-variable functions, I

1.1. **Partial Derivatives+ Linear Approximation.** Compute the partial derivatives of $x^2 - y^2$ at the point $(1,1)$. Use these derivatives to estimate the value of the function at the value $(1.5, 1.5)$.
Solution: We have

$$\partial_x f(x,y) = 2x$$
$$\partial_y f(x,y) = -2y$$

The approximation is

$$\begin{aligned} f(1+.5, 1+.5) \simeq& f(1,1) + .5(\partial_x f)((1,1) + .5(\partial_y f(1,1) \\ =& 0 + .5(2) + .5(-2) \\ =& 0 \end{aligned}$$

This is actually the value of $f(1.5, 1.5)$, so the approximation is not so bad.

1.2. **Partial Derivatives + Linear Approximation II.** Consider the function $\frac{xy}{\sqrt{x^2+y^2}}$. Compute the partial derivatives in both the x and y directions at 0. Estimate the value of the function at the point $(1,1)$. Why is this such a bad estimate? (You may want to draw a contour plot to help understand why this estimate is so bad?
Solution: Notice that we can take

$$\partial_x(f)(0,0) = \frac{d}{dt}(f(t,0))\bigg|_{t=0}$$

So, $\partial_x(f)(0,0) = 0$. Similarly, $\partial_y(f)(0,0) = 0$. From the example in class, we know that $f(x,y)$ is continuous at the origin, and so $f(0,0) = 0$. Therefore, our approximation would say that $f(1,1) = 0$. However, $f(1,1) = \frac{1}{\sqrt{2}}$.
The reason this approximation is so bad is because derivatives of this function at 0 only exist in the x and y direction. Notice, for instance, along the $x = y$ line parameterized by $\langle t, t \rangle$,

$$f(t,t) = t^2/\sqrt{2t^2} = |t|/\sqrt{2}.$$

This isn't even differentiable at $t = 0$. So along the $x = y$ line, there is no derivative. Therefore, it is not reasonable to hope that the function can be approximated in the $x = y$ direction by derivatives in the x and y direction.

1.3. **Limits.** Show that these functions have limit values defined at the origin, or do not have a limit at the origin.

$$\frac{x+y}{x^2+y^2} \qquad \frac{x+y}{\sqrt{x^2+y^2}} \qquad \frac{\sin x+\sin y}{x+y}$$

Solution: The first one is not continuous at the origin. Take $y=0$. Then this is $x/x^2=1/x$ which is very not continuous at the origin.
The second one is not continuous at the origin. Notice that if we approach along $x=-y$, we get 0, and if we approach along $x=y$, we get ± 1 (depending if we approach from the first quadrent or the 3rd quadrant.)