WORKSHEET, FEB 3

0.1. **Lines and Planes.**

(1) Find a plane P that contains the origin and the line ℓ defined by

$$x = t + 1$$
$$y = -1$$
$$z = -t$$

Solution: We need to find 3 points on the plane. The points we can use this example wil be given by the origin, and 2 points on the line, $\ell(0) = (1, -1, 0)$ and $\ell(1) = (2, -1, -1)$. Then we have two vectors in this plane, given by $\langle 1, -1, 0\rangle = (1, -1, 0) - (0, 0, 0)$ and similarly $\langle 2, -1, -1\rangle$. The normal to the plane can be found by taking the cross product of these vectors, giving us $\vec{N} = \langle 1, 1, 1\rangle$. So, the equation for the plane is given by

$$(1)x + (1)y + (1)z = c$$

for some constant c that we need to find. By checking this equation on a single point (say, the origin,) we get that $c = 0$. So the equation of the plane is

$$(1)x + (1)y + (1)z = 0$$

(2) Using the same line and plane from the above example, find a new line ℓ_1 which
- Contains the origin
- Is perpendicular to our original line ℓ.

Solution: The vector from the origin to the point $\ell(t)$ is simply $\langle t + 1, -1, -t\rangle$. We want to know when this is perpendicular to the direction that ℓ travels in, which is $\langle 1, 0, -1\rangle$. So we want to find the value of t solving

$$0 = \langle t + 1, 2t - 1, -t\rangle \cdot \langle 1, 0, -1\rangle = t + 1 + 4t - 2 +$$

which is when $t = \frac{1}{2}$. So the direction that the line travels in is

$$\ell(1/6) = (3/2, -1, -1/2)$$

We can scale this direction to something easier to look at, so the direction the new line travels in is $\vec{v} = \langle 3, -2, -1\rangle$.

Slick Way, first solving problem 3 The direction that this new line travels in can be found by taking the cross product of the normal vector of the plane, and the direction that ℓ travels in. We compute

$$\langle 1, 2, -1\rangle \times \langle 1, 1, 1\rangle = \langle 3, -2, -1\rangle$$

The equation of a line is given by $\ell_1(t) = p_0 + \vec{v}t$, where $\vec{v}$ is the direction of travel and p_0 is some initial point. By assumption, the new line contains the origin, so the equation of this new line is

$$x_1(t) = 3t$$
$$x_2(t) = -2t$$
$$x_3(t) = -1t$$

(3) Why is it that ℓ_1 is contained in the plane P?
Solution: Because it contains 2 points from P– the origin, and a point on the line ℓ. If a plane and a line agree on 2 points, then the line is contained inside the plane.

(4) Find a unit vector perpendicular to the plane $3x + y - z = 2$.
Solution:A perpendicular vector is given by the coefficients of the equation. So we know that a perpendicular vector is $\vec{N} = \langle 3, 1, -1\rangle$. To get a unit vector, we scale this vector by $\frac{1}{|\vec{N}}$m which gives us the unit vector $\langle 3/\sqrt{1}1, 1/\sqrt{1}1, -1/\sqrt{1}1\rangle$.

(5) Show that if P_1, P_2, P_3 all contain a common line, then the normal vectors $\vec{n}_1, \vec{n}_2$, and $\vec{n}_3$ to these planes all lie in the same plane.
Solution: Let $\vec{v}$ be the direction of the common line. Then $\vec{n}_1 \cdot \vec{v} = \vec{n}_2 \cdot \vec{v} = \vec{n}_3 \cdot \vec{v} = 0$. Therefore, they are contained in the normal plane to $\vec{v}$.

(6) Describe an algorithem which finds the minimal distance between 2 lines (which does not involve taking a derivative!) **Solution:**Let the first line be given by $t\vec{v}_1 + p_1$, and the second by $t\vec{v}_2 + p_2$. Then the shortest distance between these two lines is a line which is perpendicular to both of them. So we want to find a new line $t\vec{v}_3 + p_3$ which intersects both of our other lines perpendicularly. Therefore $\vec{v}_3 = \vec{v}_1 \times \vec{v}_2$. We now look at the *plane swept out by the lines travelling in direction* $\vec{v}_3$ *which have an initial point on the first line* $t\vec{v}_1 + p_1$. This plane has normal direction

$$\vec{N} = \vec{v}_1 \times \vec{v}_3$$

and contains the point p_1. Once we have found the equation of this plane, whe just need to check where it intersects the line $t\vec{v}_2 + p_2$. Then we will know the point p_3.