0.1. Cross Product of vectors.

(1) Find 2 vectors perpendicular to both $\vec{v}$ and $\vec{u}$:

$$\vec{u} = \langle 1, 3, 2\rangle \qquad \vec{v} = \langle -1, 0, 3\rangle$$

Solution: One possibility is the cross product of $\vec{v}$ and $\vec{u}$.

$$\begin{aligned}\langle 1,3,2\rangle \times \langle -1,0,3\rangle =& \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 3 & 2 \\ -1 & 0 & 3 \end{vmatrix} \\ =& \hat{i}\begin{vmatrix} 3 & 2 \\ 0 & 3 \end{vmatrix} - \hat{j}\begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix} + \hat{k}\begin{vmatrix} 1 & 3 \\ -1 & 0 \end{vmatrix} \\ =& (9-0)\hat{i} - (3-(-2))\hat{j} + (0-(-3))\hat{k} = \qquad \langle 9, -5, 3\rangle \end{aligned}$$

Another vector can be found by taking any scaling of this vector, such as $\langle -9, 5, -3\rangle$.

(2) Find the area of the triangle with edges given by vectors

$$\langle 1, 1, 1\rangle, \langle 2, -1, 0\rangle$$

Solution: The area of such a triangle is $|\vec{u} \times \vec{v}|/2$. This gives us

$$\begin{aligned}\langle 1,1,1\rangle \times \langle 2,-1,0\rangle =& \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ 2 & -1 & 0 \end{vmatrix} \\ =& \hat{i}\begin{vmatrix} 1 & 1 \\ -1 & 0 \end{vmatrix} - \hat{j}\begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix} + \hat{k}\begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} \\ =& (1-0)\hat{i} - (0-2)\hat{j} + (-1-2)\hat{k} = \qquad \langle 1, 2, -3\rangle \end{aligned}$$

The length of this vector is $\sqrt{1^2 + 2^2 + (-3)^2} = \sqrt{14}$, so the area of the triangle is $\sqrt{14}/2$.

(3) For which value of a is the following cross product the zero vector?

$$\langle 2, -2, 3\rangle \times \langle 1, -1, a\rangle$$

Solution: We have that if $\vec{v} = s\vec{u}$, then $\vec{v} \times \vec{u} = 0$. When $a = \frac{3}{2}$, the first vector is a multiple of the second, and so the cross product is zero.

(4) When in general is $\vec{v} \times \vec{u}$ 0?
Solution: We have that if $\vec{v} = s\vec{u}$, then $\vec{v} \times \vec{u} = 0$.

(5) Explain that if $\vec{v}, \vec{u}$ satisfie

$$|\vec{u} \times \vec{v}| = \vec{u} \cdot \vec{v} = 0$$

that one of $\vec{v}$ or $\vec{u}$ must be the zero vector.