Discussion Problems, Parametric Curves II

(1) *10.1.37* Consider the parametric equations

$$x_1(t) = e^t \quad y_1(t) = e^{-2t}$$

$$x_2(s) = \cos(s) \quad y_2(s) = \sec^2 s$$

By converting each of these curves to a cartesian equation, show they represent the same curve.
Solution: For the first equation, we have

$$y_1(t) = e^{-2t} = (e^t)^-2 = x_1(t)^{-2}$$

For the second equation, we have

$$y_2(s) = \sec^2 s = (\cos s)^{-2} = x_2(s)^{-2}$$

Find an equation for t in terms of s that reparameterizes first equation into the second. **Solution:**Let $t = \ln(\cos s)$.

(2) Given two curves parametric equations, we can consider their *intersection points* and their *collision points*.

- An intersection point is where the two equations have the same x and y values, but possibly at different times.
- A collision point is where the two equations have the same x and y values at the same time.

Consider the curves given by

$$x_1 = 3\sin t \quad y_1 = 2\cos t$$

$$x_2 = t \quad y_2 = 0$$

Find any intersection or collision points that exist for these 2 equations. **Solution:**Let's first look for intersection points. That means, we are searching for s and t so that

$$3\sin t = s$$
$$2\cos t = 0$$

The second equation is easier to work with. If there is an intersection point, then $2\cos t = 0$, which means that $t = \pi/2 + n\pi$. Plugging this into the first equation, we have $3\sin(\pi/2 + n) = s$, which means that $s = \pm 3$. So, we have to intersections points, one at $(3, 0)$ and the other at $(0, 3)$.
Check! You can verify this geometrically by plotting out the two curves. The first one is an ellipse centered at the origin, and the other plots out the x axis. We expect these shapes to intersect at two points.

Can you come up with a different parameterization of the second equation that turns the intersection points into collision points?
Solution: There are many ways to do this, but we would like it so that at time $\pm\pi/2$ the second curve is at $(\pm 3, 0)$. One possible way to reparameterize this is

$$x_{new}(t) = 6/\pi t \quad y_{new} = 0.$$

Another way that I saw in class (although it does not draw out all of the second curve, depending on the bounds for t) is $x_{new} = 3\sin t$. Is it always possible to do this? That is, given 2 parametric curves, can you always reparameterize them in such a way that every intersection point is a collision point?

Solution: This one is tricky and conceptual... come see me during office hours to talk about this one.

(3) Consider the curve

$$x(t) = \cos(t) + \sin(t)$$
$$y(t) = \sin(3t)$$

When does this curve achieve a maximal x value?

Find the equation for a vertical line which is tangent to this curve.

By finding the maximum x values and y values, you can get a better idea of what a curve looks like. Graph the curve using this information.