"How do I go Backward"

In class, the question was asked

> Given a parametric equation, $(x(t), y(t))$, how do I make it go backward?

I realize where the tricky spot in the explanation. You need the important additional information of the *interval* which the parameter t can run between. Let's say that $t_{init} \leq t \leq t_{fin}$.
So, we want to create a new set of parametric equations (let's call them $(x_r(s), y_r(s)$, where r is for *reverse*) which walk along the same curve as the original equations, but in the backward order. The way we will create these new equations is to *reparameterize* the variable t in the old equations. This means that we're going to find a function $f(s) = t$ and set

$$x_r(s) = x(f(s))$$
$$y_r(s) = y(f(s))..$$

We'll have to pick the function $f(s)$ carefully so that the new curve runs backward along the old curve. Let's say ahead of time that the parameter s runs from 0 to 1. If the new curve runs in the opposite direction, then $(x_r(0), y_r(0))$ should be the *last point* of the original curve,– which is $x(t_{fin})$. This means,

$$f(0) = \text{The largest value that } t \text{ can take} = t_{fin}$$

Similarly, the last value that s takes should correspond to the first value that t takes.

$$f(1) = \text{The smallest value that } t \text{ can take} = t_{init}$$

We can always just take the function $f(s)$ to be

$$f(s) = (1-s) \cdot t_{fin} + s \cdot t_{init},$$

but there are other examples of $f(s)$ that work.

> You should think about why this particular function works...

If your parameter s is suppose to run between something other than 0 to 1, you may have to do something more tricky! Let's look at a few examples.

0.1. **Example 1.** Consider the line segment

$$x(t) = 3t + 1 \quad y(t) = 2t + 2$$

where $0 \leq t \leq 1$. We want to find a parametrization for this line that draws it in the reverse order.
Letting $f(s) = (1-s) \cdot 1 + s \cdot 0 = (1-s)$, we get

$$x_r(s) = x(f(s)) = 3(1-s) + 1$$
$$y_r(s) = y(f(s)) = 2(1-s) + 1$$

0.2. **Example 2.** The second quarter circle is drawn by

$$x(t) = \cos(t) \quad y(t) = \sin(t)$$

where t runs between $\pi/2$ and π. Parameterize it backwards.
In this case, we have

$$f(s) = (1-s) \cdot \pi + s \cdot \pi/2 = \pi - \pi/2s$$

Then we have

$$x_r(s) = x(f(s)) = \cos(\pi - \pi/2s)$$
$$y_r(s) = y(f(s)) = \sin(\pi - \pi/2s)$$

This can be simplified by applying trig identities to

$$x_r(s) = -\cos(\pi/2s)$$
$$y_r(s) = \sin(\pi/2s)$$

0.3. **Example 3, the one that I got caught up on.** The funny line is drawn by

$$x(t) = \sin^2(t) \quad , y(t) = \cos^2(t)$$

where t runs between $-\pi$ and π. Parameterize it backwards.
In this case, I'm going to do something different. Instead of letting s run from $0, 1$, let's have

$$-\pi \leq s \leq \pi.$$

Then we get that running t backwards is the same as letting

$$t = -s.$$

This substitution gives us

$$x_r(s) = \sin^2(-s)$$
$$y_r(s) = \cos^2(-s)$$

This can be simplified to

$$x_r(s) = \sin^2(s)$$
$$y_r(s) = \cos^2(s)$$

Wait! This looks the same as the curve being parameterized forwards. This is ok; the parametric equation draws a line segment going forward, then going backward to its starting position. The path taken walking along backward is the same taken as the path taken walking along the curve forward.

Things to think about

- Where do the equations $(x_r(s), y_r(s))$ and $(x(t), y(t))$ intersect each other?
- Suppose that t runs from 0 to 1. Prove that $(x_r(t), y_r(t))$ and $(x(t), y(t))$ have a collision point.