1. Given the matrix

$$A = \begin{pmatrix} 1 & -1 & 1 \\ -1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

   (a) Find the eigenvalues and eigenvectors to A

   (b) Find a set of orthonormal eigenvectors to A

   (c) Find a matrix S such that $SAS^\top = D$, where D is diagonal.

   (d) Compute A^6.

   Solution

   (a) For the first part, you just take the characteristic polynomial as usual. Look at

$$\begin{aligned}
\det(A - \lambda\,\mathrm{id}) =& \det\left(\begin{pmatrix} 1 & -1 & 1 \\ -1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix} - \lambda\,\mathrm{id}\right) \\
=& \det\left(\begin{pmatrix} 1-\lambda & -1 & 1 \\ -1 & 0-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{pmatrix}\right) \\
=&(1-\lambda)^2\lambda + (-1)(1)(1) + (1)(-1)(1) - ((1-\lambda) + (1-\lambda) + (-\lambda)) \\
=&(-\lambda + 2\lambda^2 - \lambda^3) - 4 + 3\lambda \\
=& -\lambda^3 + 2\lambda^2 + 2\lambda - 4
\end{aligned}$$

   Buy guessing a root that divides the last term 4, you find that 2 is a root of this polynomial. After doing polynomial long division you get

$$= -(\lambda - 2)(\lambda^2 - 2)$$

   which means that the other eigenvalues are $\sqrt{2}$ Let's start with the first eigenvalue.

   i. With $\lambda = 1$, you get that

$$\begin{pmatrix} -1 & -1 & 1 \\ -1 & -2 & 1 \\ 1 & 1 & -1 \end{pmatrix} \vec{v} = 0$$

   Which you can solve by your favorite method to see that $(1, 0, 1)$ is a solution.

   ii. With $\lambda = -\sqrt{2}$, you get that

$$\begin{pmatrix} 1-\sqrt{2} & -1 & 1 \\ -1 & -\sqrt{2} & 1 \\ 1 & 1 & 1-\sqrt{2} \end{pmatrix} \vec{v} = 0$$

   You know that the second eigenvector needs to be perpendicular to the first, and is therefore of the form $(-1, \alpha, 1)$ for some value of α. By checking the second row, you might have guessed $(-1, \sqrt{2}, 1)$. And you would have been right.

   iii. At this point, you're probably pretty tired of finding eigenvalues.

$$\begin{pmatrix} 1+\sqrt{2} & -1 & 1 \\ -1 & \sqrt{2} & 1 \\ 1 & 1 & 1+\sqrt{2} \end{pmatrix} \vec{v} = 0$$

   By again checking the second row, you may have guessed $(-1, -\sqrt{2}, 1)$. Good guess! You now have 3 eigenvectors and 3 eigenvalues.

(b) The ones we found also happened to be orthonormal. Hoorah.

(c) Since we already have an orthonormal basis, we can use

$$S = \begin{pmatrix} 1 & -1 & 1 \\ 0 & -\sqrt{2} & \sqrt{2} \\ 1 & 1 & 1 \end{pmatrix}$$

Of course, D is given by the eigenvalues

$$D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & -\sqrt{2} \end{pmatrix}$$

(d) To do this, you can take

$$SA^6S^\top$$

2. Find the least squares fit quadratic polynomial to the following data

$$(-1,0)$$
$$(0,0)$$
$$(1,0)$$
$$(2,20)$$

Solution: Like before, you are going to construct a map from the space of quadratic polynomials to $\mathbb{R}^4$. How do you construct this map?

$$\begin{aligned} A : \mathbb{P}^2 &\to \mathbb{R}^4 \\ f(x) &\mapsto (f(-1), f(0), f(1), f(2)) \end{aligned}$$

Write A out as a matrix in the basis $\{1, x, x^2\}$

$$A = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{pmatrix}$$

You then can just solve the equation $(A^\top A)^{-1}A^\top(0,0,0,20)$ The solution is $(-3,1,5)$, so the polynomial is $5x^2 + x - 3$

3. True or False!

(a) If $\vec{v}$ is an eigenvector for T, and $TS = ST$, then $\vec{v}$ is an eigenvector for S.

(b) The formula $\left\langle \begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} c \\ d \end{pmatrix} \right\rangle = tr \begin{pmatrix} a & c \\ b & d \end{pmatrix}$ defines an inner product of $\mathbb{R}^2$

(c) Suppose A and B are orthogonal matrices. Then $A + B$ is an orthogonal matrix.

(d) If A is orthogonal, and B is orthogonal, then AB is an orthogonal matrix.

(e) Suppose A is orthogonal, and B is orthogonal. Then $A + B$ is invertible.

(f) Let $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ be a orthonormal basis for $\mathbb{R}^4$. Then

$$\left\{ \frac{\sqrt{2}}{2}\vec{v}_1 - \frac{\sqrt{2}}{2}\vec{v}_2, \frac{\sqrt{2}}{2}\vec{v}_1 + \frac{\sqrt{2}}{2}\vec{v}_2, -\vec{v}_3, \vec{v}_4 \right\}$$

is also an orthonormal basis for $\mathbb{R}^4$.

Solution:

(a) This is false! Let B be rotation by 90 degrees, and let A be the identity. Then $AB = BA$, but B has no eigenvectors. But everything is an eigenvector of A.

(b) False! it does not. Take

$$\left\langle \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\rangle = 0$$

(c) False! Take id and $-$ id.

(d) True! Recall orthogonal means that the transpose is the same as the inverse. Then

$$(AS)^\top = S^\top A^\top = S^{-1}A^{-1} = (AS)^{-1}$$

(e) False! Take id and $-$ id.

(f) True! You can check using the rules for the dot product that this basis is again orthonormal.