Practice Final 1

1. True or False!

   (a) Suppose that $B^{-1}AB$ is diagonal. Then A^2 is diagonalizable.

   (b) If $\langle v, w\rangle = 0$, $\langle w, u\rangle = 0$, then $\langle v, u\rangle = 0$.

   (c) If v is an eigenvector of A and A is invertible, then it is also an eigenvector for A^{-1}

   (d) If λ is an eigenvalue of A and A is invertible, then $\frac{1}{\lambda}$ is also an eigenvalue of A^{-1}.

   (e) Solutions to the wave equation $\frac{\partial^2}{\partial t^2}u(x,t) = \frac{\partial^2}{\partial x^2}u(x,t)$ are of the form $u(t,x) = T(t)X(x)$.

   (f) Suppose that $y_1(t)$ and $y_2(t)$ are linearly independent functions. Then the Wronskian $W[y_1, y_2](t) \neq 0$ for all t.

   (g) Suppose that A and B are not orthogonal. Then AB is not orthogonal.

   (h) The number of solutions to a k-th order differential equation is k.

   (i) Suppose the rank of A is 4, and the rank of B is 3. Then the largest the rank of AB can be is 4.

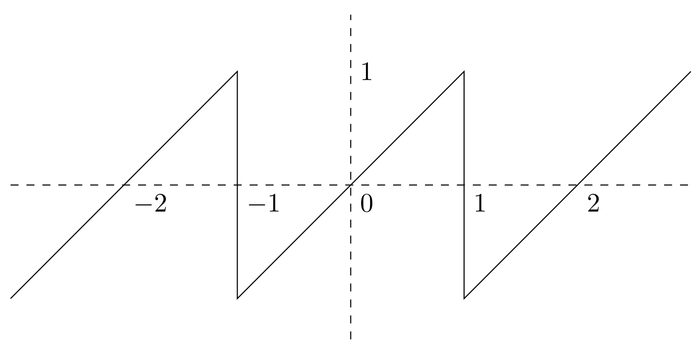
Solution 1

- True! If $B^{-1}AB$ is diagonal, then so is $B^{-1}ABB^{-1}AB = B^{-1}A^2B$
- False. Take for instance, $w = 0$, and $v = u \neq 0$.
- True!
- True!
- No, but they can be written as a sum of such things via the Fourier transform.
- No, this is only true when we know that y_1 and y_2 are solutions to a Linear ODE
- No, take $B + A^{-1}$. Then $AB = I$, but neither A nor B are orthogonal.
- No, the dimension of the solutions to a k-th order differential equation is k.
- No, the largest the rank can be is 3.

2. The sawtooth wave is given by the piecewise function:

$$f(x) = x - 2\left\lfloor \frac{1}{2}(x+1)\right\rfloor$$

where $\lfloor x\rfloor$ is the value of x rounded down to the nearest integer. Pictorially, it looks like this:



Expand this out as a Fourier Series.

Solution 2 We need to compute the Fourier coefficients. As the function is odd, we only need to work out the ones in front of the $\sin\left(\frac{n\pi}{L}d\right)$. Since $L = 1$, we have that

$$f(x) = \sum_{n=1}^{\infty} a_n \sin(n\pi x)$$

where $a_n = \int_{-1}^{1} f(x)\sin(n\pi x)dx$. On the interval -1 to 1, the function $f(x) = x$,so we have so we get

$$\begin{aligned} a_n &= \int_{-1}^{1} f(x)\sin(n\pi x)dx \\ &= \int_{-1}^{1} x\sin(n\pi x)dx \end{aligned}$$

By Oddness of both x and sin

$$=2\int_{0}^{1} x\sin(n\pi x)dx$$

Integrating by parts

$$=2\left(\frac{-x}{n\pi}(\cos(n\pi x)\Big|_0^1 - \int_0^1 (\frac{-1}{n\pi}\cos(n\pi x))dx\right)$$

Because the integral of the right term is always 0,

$$=\frac{(-1)^{n+1}2}{\pi n}$$

This gives us that the Fourier series is

$$f(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}2}{\pi n}\sin(n\pi x)$$

3. Solve the differential equation $y' = At$, where

$$A = \begin{pmatrix} 4 & -1 & -2 \\ 2 & 1 & -2 \\ 5 & 0 & -3 \end{pmatrix}$$

Solution 3 The first thing to do is to find the eigenvectors and eigenvalues to A. We have that

$$\begin{aligned} \det(A - \lambda I) &= \det\begin{pmatrix} 4-t & -1 & -2 \\ 2 & 1-t & -2 \\ 5 & 0 & -3-t \end{pmatrix} \\ &= -x^3 + 2x^2 - x + 2 \\ &= (2-x)(x^2+1) \end{aligned}$$

This gives us that the eigenvalues are $2, \pm i$. The eigenvector associated to 2 is

$$v = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

while one of the complex eigenvectors w for $t = i$ and u for $t = -i$ are

$$w = \begin{pmatrix} 3+i \\ 3+i \\ 5 \end{pmatrix} \qquad\qquad u = \begin{pmatrix} 3-i \\ 3-i \\ 5 \end{pmatrix}$$

This gives us linearly independent solutions:

$$\begin{aligned} y_1 &= * \, e^{2t} v \\ z_1 &= e^{it} w \\ z_2 &= e^{-it} u \end{aligned}$$

By taking a linear combination of the latter complex solutions we get

$$\begin{aligned} y_2 &= \frac{z_1 + z_2}{2} \\ &= \frac{1}{2}\left((\cos t + i \sin t) \begin{pmatrix} 3+i \\ 3+i \\ 5 \end{pmatrix} + (\cos t - i \sin t) \begin{pmatrix} 3-i \\ 3-i \\ 5 \end{pmatrix} \right) \\ &= \cos t \begin{pmatrix} 3 \\ 3 \\ 5 \end{pmatrix} - \sin t \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \end{aligned}$$

Likewise, by letting $y_3 = \frac{(z_1 - z_2)}{2i}$ we have

$$y_3 = \sin t \begin{pmatrix} 3 \\ 3 \\ 5 \end{pmatrix} + \cos t \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

The general solutions are then of the form

$$y = c_1 y_1 + c_2 y_2 + c_3 y_3$$

4. Diagonalize the following matrix:

$$\begin{pmatrix} 4 & 1 & -1 \\ 4 & 0 & 2 \\ 2 & -1 & 3 \end{pmatrix}$$

Solution 4 We start by calculating the characteristic polynomial:

$$\det \begin{pmatrix} 4-t & 1 & -1 \\ 4 & 0-t & 2 \\ 2 & -1 & 3-t \end{pmatrix} = -t^3 + 7t^2 - 12t + 4$$

We have that 2 is a root of this polynomial, so it factors as

$$-t^3 + 7t^2 - 12t + 4 = (2-x)(x^2 - 5x + 2)$$

This gives us that the roots are 2, $\frac{5 \pm \sqrt{17}}{2}$. Let's calculate the eigenvectors associated to each one of these.

- For the first eigenvalue, we have to find something in the kernel of

$$A - 2I = \begin{pmatrix} 2 & 1 & -1 \\ 4 & -2 & 2 \\ 2 & -1 & 1 \end{pmatrix}$$

Since the second and third columns are linearly dependent, we have that the first eigenvector is

$$\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

- For the second eigenvector, we have to find something in the kernel of

$$A - \frac{5+\sqrt{17}}{2} I = \begin{pmatrix} 4 - \frac{5+\sqrt{17}}{2} & 1 & -1 \\ 4 & -\frac{5+\sqrt{17}}{2} & 2 \\ 2 & -1 & 3 - \frac{5+\sqrt{17}}{2} \end{pmatrix}$$
$$= \begin{pmatrix} \frac{3-\sqrt{17}}{2} & 1 & -1 \\ 4 & -\frac{5+\sqrt{17}}{2} & 2 \\ 2 & -1 & \frac{1-\sqrt{17}}{2} \end{pmatrix}$$

It doesn't look like there are any fancy tricks that are going to get us out of this one! At this point, it is actually easier to solve by substitution than by row reduction. This is because you have the choice of a free variable in your computation! Let $v_2 = (a, b, c)$ be in the kernel of this matrix. Because we have a free variable, we can choose a at will; I choose it to the conjugate of the left top corner of the matrix, that is $a = \frac{3+\sqrt{17}}{2}$. That way the first line of the matrix tells us

$$-2 + 1b - 1c = 0$$

The second line now reads

$$4\left(\frac{3+\sqrt{17}}{2}\right) - \left(\frac{5+\sqrt{17}}{2}\right) b + 2c = 0$$

Subbing in $b = c + 2$ in yields

$$4\left(\frac{3+\sqrt{17}}{2}\right) - \left(\frac{5+\sqrt{17}}{2}\right)(c+2) + 2c = 0$$
$$c = 2$$

Which tells us that

$$b = 4$$

This gives us our second eigenvector to be:

$$v_2 = \begin{pmatrix} \frac{3+\sqrt{17}}{2} \\ 2 \\ 4 \end{pmatrix}$$

- For the third eigenvector, you can repeat an argument similar to that above to get

$$v_3 = \begin{pmatrix} \frac{3-\sqrt{17}}{2} \\ 2 \\ 4 \end{pmatrix}$$

Protip: notice that v_2 and v_3 are conjugate in the sense that the $+\sqrt{17}$ is now a $-\sqrt{17}$. This type of phenomenon is true in general, but is pretty hard to show. In any case, it is a sign that we are on the right track.

The change of basis matrix is now given by

$$B = \begin{pmatrix} 0hg & \frac{3+\sqrt{17}}{2} & \frac{3-\sqrt{17}}{2} \\ 1 & 4 & 4 \\ 1 & 2 & 2 \end{pmatrix}$$