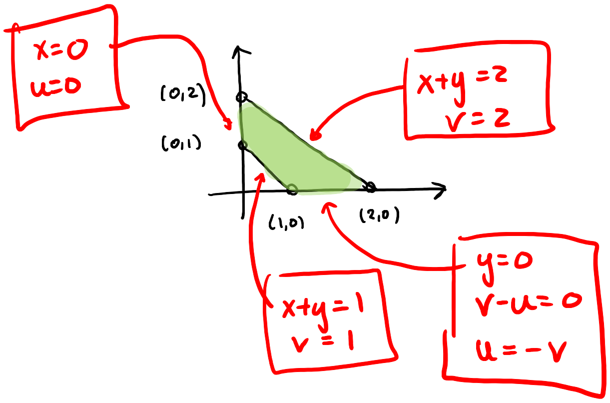
Quiz, Oct. 22

Name:

Jacobian (6). Set up, but do not compute, an integral which computes the integral

$$\iint_D (x+y)\, dx dy$$

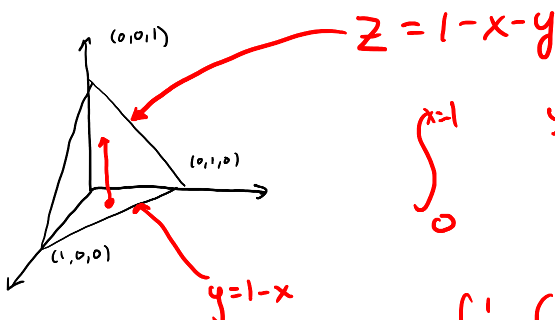
over the following domain D, using the coordinate change $u = x$ and $v = x + y$.



$$\int_{v=1}^{v=2}\int_{u=-v}^{u=0} v \; du\, dv.$$

$$\boxed{\text{Jacobian} \quad \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}^{-1} = |1|^{-1} = 1}$$

Triple Integral (7). Compute the integral of the function $f(x, y, z) = x$ over the region in the first octant bounded by $x + y + z \leq 1$.



$$\int_0^{x=1}\int_0^{y=1-x}\int_0^{z=1-x-y} x \; dz\, dy\, dx$$

$$= \int_0^1 x \int_0^{1-x} (1-x-y)\; dy\, dx$$

$$= \int_0^1 x\left(y - xy - \tfrac{1}{2}y^2\right)\Big|_{y=0}^{y=1-x} dx$$

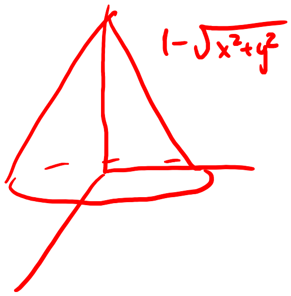
$$= \int_0^1 x\left(1-x - x(1-x) - \tfrac{1}{2}(1-x)^2\right) d$$

$$= \int_0^1 x\left(1-x-x+x^2 - \tfrac{1}{2} + x - \tfrac{1}{2}x^2\right) = \int_0^1 x\left(\tfrac{1}{2} - x + \tfrac{1}{2}x^2\right) dx$$

$$= \int_0^1 \frac{x}{2} - x^2 + \frac{1}{2}x^3\, dx = \frac{x}{4} - \frac{x^3}{3} + \frac{1}{8}x^4$$

$$= \frac{3}{8} - \frac{1}{3} = 1/24.$$

Triple Integral (7). A cone is given by the region between $0 \leq z \leq 1 - \sqrt{x^2 + y^2}$. Set up an integral computing the volume of the cone using cylindrical coordinates. Your answer should be in the form of a triple integral. It may help to draw a picture first.



$0 \leq z \leq 1-r$

$$\boxed{\int_0^{2\pi} \int_{r=0}^{r=1} \int_{z=0}^{z=1-r} 1 \; r \, dz \, dr \, d\theta}$$

Bonus Problem. *Worth no points!* A function $f(x, y)$ is called *harmonic* if $f_{xx} + f_{yy} = 0$. Show that if a function is harmonic on a compact region R, that the maximum of f must occur on the boundary of the region.

Harmonic functions are time-constant solutions to the heat equation

$$\frac{\partial}{\partial t} f(t, x, y) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) f(t, x, y)$$

which describes how a function $f(t, x, y)$ describing the tempurature at a point (x, y) at time t behaves. Why does it make physical sense that if we have a time-constant solution to this equation, the maximum must be achieved at the boundary of the region?