Practice Midterm

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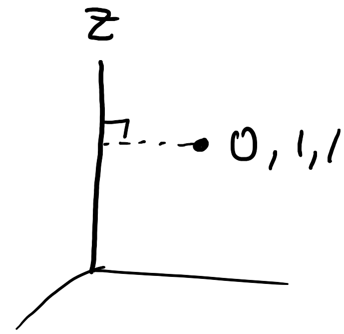
Lines and Planes.

(a) The line $\ell(t)$ which is perpendicular to the z axis and goes through the point $(0, 1, 1)$.
(b) Find the equation of a plane containing both the z axis and the point $(0, 1, 1)$.
(c) Show that the line $\ell(t)$ is contained within the plane.



Since line is $\perp$ to z axis, if line is

$$\underline{\vec{r}(t) = \vec{v}t + p_0}$$

We have that $\vec{v} \cdot \langle 0,0,1\rangle = 0$

so $\vec{v} = \langle a, b, 0\rangle$.

Let $p_0 = \langle 0,1,1\rangle$, a point on our line. Then.

$$t \cdot \langle a, b, 0\rangle + \langle 1,0,0\rangle = \langle 0,0,c\rangle$$

where $(0,0,c)$ is point on z-axis. So, $a \neq 0$, $b = 0$.

$$\vec{r}(t) = \langle 0, t, 1\rangle.$$

To find plane, know that these two vectors are $\parallel$ to the plane. So.

$$\boxed{\begin{array}{l} z \text{ axis} = \langle 0,0,1\rangle \\ \vec{r} \text{ direction} = \langle 0,1,0\rangle \end{array}}$$

$$\hat{n} = \langle 0,0,1\rangle \times \langle 0,1,0\rangle = \langle 1,0,0\rangle$$

So plane is $1x + 0y + 0z + d = 0$

plugging in point .1) gives $d = 0$.

So, plane is $x = 0$

Plugging in
$x(t) = 0$
$y(t) = t$
$z(t) = 1$
into $x = 0$.

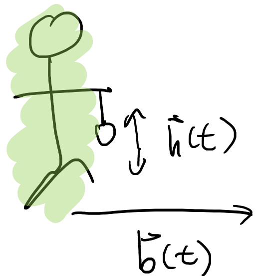
$x(t) = 0$ so line in within plane.

Parameteric Curves. A boy starts at the origin at time $t = 0$ with a velocity of $\langle 1, 1\rangle$. As walks, he spins a yo-yo. From time -2 to time 2, the altitude of the yo-yo can be described by $t^2 + 1$.

(a) Give a parameterization for the position of the yoyo for times $-2 \le t \le 2$.

(b) Where does the yoyo maximize its speed?

(c) Compute the distance that the yoyo travels between $t = -2$ and $t = 2$.



yoyo to earth boy's position

$$\text{yoyo} = \vec{r}(t) = \vec{h}(t) + \vec{b}(t).$$

$$\vec{h}(t) = \langle 0, 0, t^2+1\rangle$$

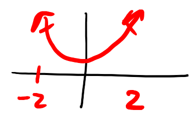
$$\vec{b}(t) = \langle 1, 1, 0\rangle \cdot t$$

$$\boxed{\vec{r}(t) = \langle t, t, t^2+1\rangle.}$$

$$\vec{v}(t) = \vec{r}\,'(t) = \langle 1, 1, 2t\rangle$$

Speed. $= |\vec{v}(t)| = \sqrt{1^2+1^2+(2t)^2} = \sqrt{2+4t^2}$.

Maximize $s^2 = 2+4t^2$



maximized on $[-2, 2]$ when $t = \pm 2$.

$$\text{Length} = \int_{-2}^{2} |\vec{v}(t)|\,dt = \int_{-2}^{2} \sqrt{2+4t^2}\,dt$$

$$= \int \sqrt{2}\sqrt{(1+2t^2)}\,dt \qquad \boxed{\begin{array}{l} u = \sqrt{2}t \\ du = \sqrt{2}dt \end{array}}$$

$$\rightarrow \boxed{= \int \sqrt{2}\sqrt{1+u^2}}\ \sqrt{2}du = 2\int \sqrt{1+u^2}\,du$$

$$= 2\int \sqrt{1+\tan^2\theta}\ \sec^2\theta\,d\theta \qquad \boxed{\begin{array}{l} u = \tan\theta \\ d\theta = \sec^2\theta \end{array}}$$

$$= 2\int \sqrt{\sec^2\theta}\ \sec^2\theta\,d\theta$$

$$\boxed{= 2\int \sec^3\theta\,d\theta} \qquad \cos^2 + \sin^2 = 1$$

$$= \int \sec(1+\tan^2\theta\,d\theta \qquad 1+\tan^2 = \sec^2\theta$$

$(x-y)^2+3$

Tangent Planes, Min Max. Consider the function $f(x,y) = x^2 - 2xy + y^2 + 3$.

(a) Find the tangent plane to the graph of this function $(1,1,3)$.

(b) What critical points does $f(x,y)$ have, and what type are they?

(c) Maximize the function $f(x,y)$ on the region $x^2+y^2 \le 4$.

(a) first: $f_x = 2x-2y$, $f_y = -2x+2y$

$\hat{n} = \langle f_x, f_y, -1 \rangle$

$(x,y) = (1,1)$.

$$\left.\begin{aligned} f_x\big|_{(1,1)} &= 0 \\ f_y\big|_{(1,1)} &= 0 \end{aligned}\right\} \Rightarrow \hat{n} = \langle 0, 0, 1 \rangle.$$

Tangent plane is $0x + 0y + 1z = 3$

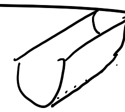
$\Rightarrow z = 3$.

(b) $f_x = 0$ when $x=y$

$f_y = 0$ when $x=y$.

Critical points along $x=y$

so, critical line minimum.



(c) Laplace multipliers!

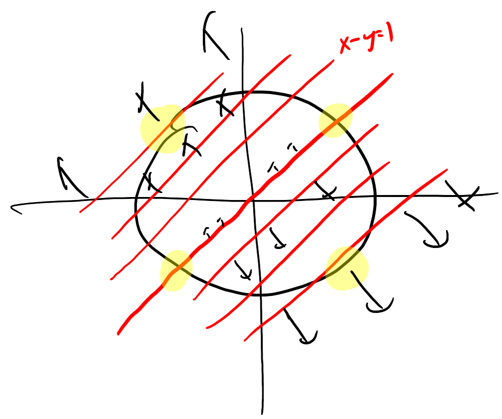
$\nabla f = \langle 2x-2y, -2x+2y \rangle$

$g = x^2+y^2$, $\nabla g = \langle 2x, 2y \rangle$

$\nabla f = \lambda \nabla g$

$\Downarrow$

Equat. $\boxed{\begin{aligned} 2x-2y &= \lambda 2x \\ -2x+2y &= \lambda 2y \\ x^2+y^2 &= 4 \end{aligned}}$



$2x - 2y = \lambda 2x$

$-2y = 2x(\lambda - 1)$

$2y = 2x(1-\lambda)$

$2y - 2x = \lambda 2y$

$2x(1-\lambda) - 2x = \lambda(2x(1-\lambda))$

$\cancel{2x} - \lambda 2x - \cancel{2x} = \lambda 2x - \lambda^2 2x$

$2x\lambda^2 - 4\lambda x = 0$

$\lambda(2x\lambda - 4x) = 0$

$\Rightarrow \lambda = 0$ or $4x = 2\lambda x$

$\lambda = 2$ $\quad$ $x = 0$

$2y = 2x$

$x^2+y^2 = 4$

$\Rightarrow x = \sqrt{2}, y = \sqrt{2}$

hence, $f(x,y) = 3$

Computations. Let $f(x,y) = x^2 + y^2$. Suppose that we know that $\vec{r}(t) = \langle x(t), y(t)\rangle$ has derivatives

$$|\vec{r}(0)| = 0$$
$$|\vec{r}'(0)| = 1$$
$$\boxed{|\vec{r}''(0)| = 0}$$

$f_x = 2x$

$f_y = 2y$

Compute

$$\frac{d^2}{dt^2} f(x(t), y(t))\Big|_{t=0}$$

Chain Rule: $f_x\, x'(t) + f_y\, y'(t) = \frac{d}{dt} f$

$2x(t)\,x'(t) + 2y(t)\,y'(t) = \frac{d}{dt} f.$

$2\langle x(t), y(t)\rangle \cdot \langle x'(t), y'(t)\rangle = \frac{d}{dt} f$

product rule for $\cdot$

$$\frac{d}{dt}\frac{d}{dt} f = 2\left(\langle x'(t), y'(t)\rangle \cdot \langle x'(t), y'(t)\rangle + \underbrace{\langle x(t), y(t)\rangle}_{\vec{r}(t)} \cdot \underbrace{\langle x''(t), y''(t)\rangle}_{\vec{r}''(t)}\right)$$

$$= 2\left(|\vec{r}'(t)|^2 + \vec{r}(t) \cdot \vec{r}''(t)\right)$$

$\vec{r}''(t) = 0$

$$= 2|\vec{r}'(0)| = 2.$$

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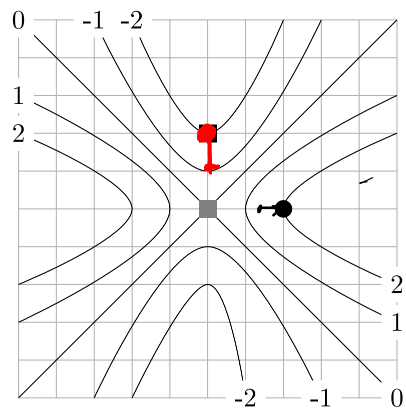
Another way $f(x(t), y(t)) = (x(t))^2 + (y(t))^2$

$$\frac{d^2}{dt^2}\left((x(t))^2 + (y(t))^2\right)$$

Contour Plots. Estimate the Gradient at each of the points marked with a square. Estimate the equation of the tangent plane at the point $(2, 0, 2)$.



Rule of thumb

Length between level curves is $\frac{1}{|\nabla f|}$

$\bullet = \langle 0, -1 \rangle$

$\blacksquare = \langle 0, 0 \rangle.$

$\bullet = \langle 1, 0 \rangle$

$f_x = 1$

$f_y = 0$

plane normal

$$\boxed{\vec{n} = \langle f_x, f_y, -1 \rangle}$$

$$= \langle 1, 0, -1 \rangle.$$

$$P_0 = (2, 0, 2)$$

$$\text{Plane eq}^{n} = \vec{n} \cdot (\vec{v} - P_0) = 0$$

$$\langle 1, 0, -1 \rangle \cdot (x-2, y-0, z-2) = 0$$

$$x - z = 0.$$

Ants! An ant travels in xy coordinates along the path $(3t, t^2)$ from time 0 to 2. It walks along a hill given whose altitude is given by $f(x, y)$. At time $t = 1$, the ant notices that their rate of altitude change is

$$\left.\frac{d}{dt} f(x(t), y(t))\right|_{t=1} = 1$$

and that their altitude is 3.

(a) Estimate the value of f at the point $(6, 3)$.

(b) Suppose further that the magnitude of the gradient $|\nabla f|_{(3,1)}$ is 2. Find the gradient $\nabla f(3, 1)$.

$\vec{v}(t) = \langle 3, 2t \rangle$

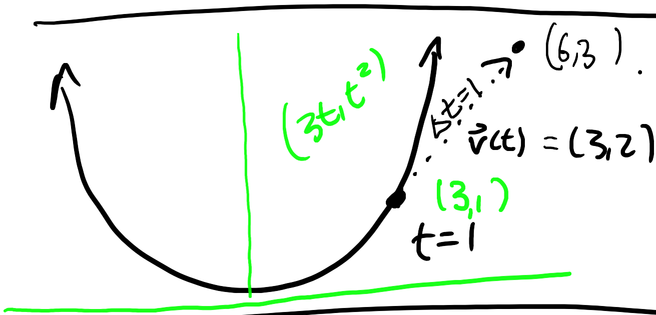
at $t=1$, $\quad \vec{r}(1) = (3,1)$

$\vec{v}(t) = (3,2)$

Notice

$(3,1) + \Delta t \cdot \vec{v}(1) = (6,3)$

$\Delta t = 1$



$$\Delta f(x(t), y(t)) \approx \Delta t \cdot \frac{df}{dt}$$

$$\approx 1 \cdot 1$$

$\Rightarrow \Delta f \approx 1 \Rightarrow f \approx 3 + 1 = 4.$

known value $\quad \Delta f$

Since magnitude of gradient is 2

and $\nabla f \cdot \vec{v}(t) = \frac{d}{dt} f(x(t), y(t)) = 1,$

chain rule

$\langle a, b \rangle \cdot \langle 3, 2 \rangle = 1$

$$\left.\boxed{\begin{array}{l} 3a + 2b = 1 \\ a^2 + b^2 = 2 \end{array}}\right\} \quad 2b = 1 - 3a$$

$4a^2 + (2b)^2 = 8$

$4a^2 + (1-3a)^2 = 8$

$4a^2 + 1 - 6a + 9a^2 = 8$

$5a^2 + 6a + 7 = 0$