Quiz, Sep 25

Name:

Directional Derivatives (8 pts). Compute the gradient of

$$f(x,y) = x^2 - y^2$$

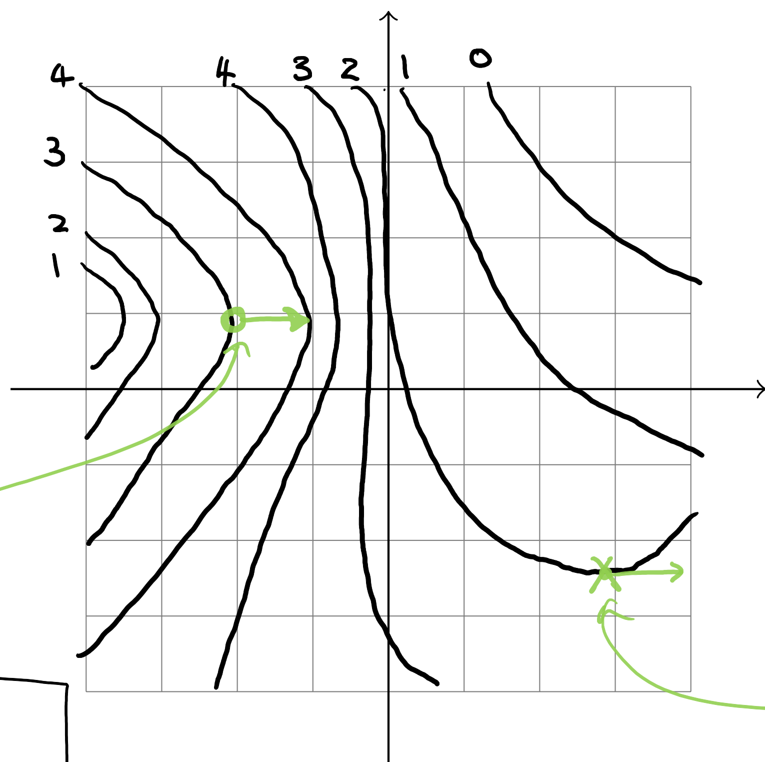
and use it to determine the directional derivative in the direction $\hat{u} = \langle \frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}} \rangle$ at the point $(1,1)$.

$$\nabla f = \langle f_x, f_y \rangle = \langle 2x, -2y \rangle$$

$$D_{\hat{u}} f \Big|_{(1,1)} = \hat{u} \cdot \nabla f|_{(1,1)} = \langle \tfrac{1}{\sqrt{2}}, -\tfrac{1}{\sqrt{2}} \rangle \cdot \langle 2, -2 \rangle = 4/\sqrt{2}$$

Derivatives and Contour Plots (6 pts).

- Mark with $\circ$ a point on this contour plot where the gradient vector is $\langle 1, 0 \rangle$. [1]
- Mark with $\times$ a point on this contour plot where the derivative in the $\hat{u} = \langle 1, 0 \rangle$ direction is 0.



Gradient is $\perp$ to Level curve, and Length $= \frac{1}{\text{distance between curves}}$

If $\hat{u}$ is tangent to level curve of f, then $D_{\hat{u}} f = 0$.

[1] Be sure to think about both the direction and magnitude of the gradient; in particular, pay attention to the values of f.

Chain Rule. (6 pts) Let $x(t) = \cos(t)$ and $y(t) = \sin(t)$. Suppose that for all t, $\frac{d}{dt} f(x(t), y(t)) = 0$. Additionally, suppose that $|\nabla f| = 1$.
Find the gradient vector at the point $(1, 0)$. [2]

$$\frac{d}{dt} f(x(t), y(t)) = f_x \, x' + f_y \, y'$$

$$= \langle f_x, f_y \rangle \cdot \langle x', y' \rangle$$

$$= \nabla f \cdot \langle -\sin t, \cos t \rangle$$

$$= \nabla f \cdot \langle 0, 1 \rangle = 0.$$

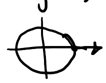
Since ∇f is perpendicular to $\langle 0, 1 \rangle$

$\Rightarrow$ ∇f is of form $\langle c, 0 \rangle$

Since $|\nabla f| = 1$, $f_x \geq 0$ $\Rightarrow$ $c = 1$.

$$\nabla f = \langle 1, 0 \rangle.$$

OR! $\frac{d}{dt} f(x(t), y(t)) = 0 \Rightarrow (x(t), y(t))$ is level curve

 and ∇f is $\perp$ to level curve.

Bonus Problem. *Worth no points!* There is a landscape whose elevation is described by the function $f(x, y)$. A hiker walks along a loop described by $\langle x(t), y(t) \rangle$. As they walk, they record the gradient vector of this hill, and use this to produce a new vector valued function

$$\vec{\kappa}(t) := \frac{\nabla f|_{(x(t),y(t))}}{\left| \nabla f|_{(x(t),y(t))} \right|}$$

(So, think of this as the function which records at time t the direction of the gradient that the hiker encounters.) The hike notices that

$$\kappa(t) = \langle \cos t, \sin t \rangle.$$

Argue why the hiker must have walked around a maxima or minima of f.

[2]Hint: Expand $\frac{d}{dt} f(x(t), y(t))$ using the chain rule. Notice that $(x(0), y(0)) = (1, 0)$.