1. Even More Derivatives

1.1. Using the Chain Rule, I. Suppose that f has a minimum at $(0,0)$. Let $\vec{r}(t) = \langle at, bt\rangle$ be a line traveling through the origin in the direction of $\langle a, b\rangle$. Show using the chain rule that

$f_x = f_y = 0$ as we are at critical point

$$\frac{d}{dt} f(\vec{r}(t))\Big|_{t=0} = 0$$

$$\frac{d^2}{dt^2} f(\vec{r}(t))\Big|_{t=0} > 0$$

$$\frac{d}{dt} f(x(t), y(t)) = f_x \frac{dx}{dt} + f_y \frac{dy}{dt}$$
$$= f_x \cdot a + f_y \cdot b$$
$$= 0.$$

$$\frac{d}{dt}\frac{d}{dt} f(x(t), y(t)) = \frac{d}{dt}\left(f_x \cdot a + f_y \cdot b\right)$$
$$= \left(f_{xx}\frac{dx}{dt} + f_{xy}\frac{dy}{dt}\right) \cdot a + \left(f_{yx}\frac{dx}{dt} + f_{yy}\frac{dy}{dt}\right) b$$
$$= f_{xx}a^2 + 2f_{xy}ab + f_{yy}b^2$$

this is always greater than 0 if $(2f_{xy}b)^2 - 4(f_{xx}f_{yy}b^2) < 0$

$\iff 4(f_{xy}^2 - f_{xx}f_{yy}) < 0 \iff$ a minimum.

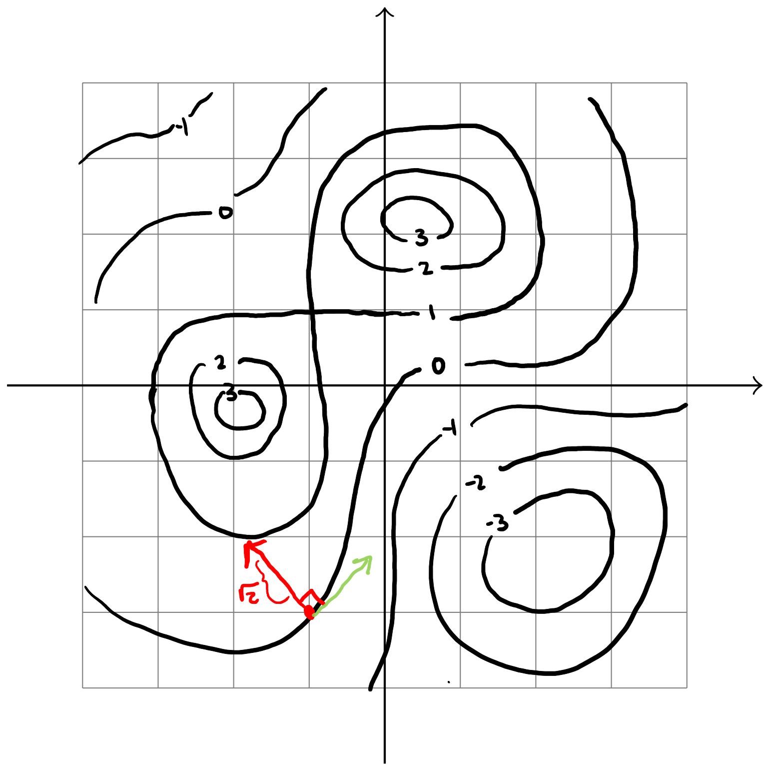
1.2. Directional Derivatives. Compute the directional derivative

$$D_{\langle \frac{\sqrt{3}}{2}, \frac{1}{2}\rangle}(x^2 + y^2)\,|_{(0,1)}$$

$$D_{\langle \frac{\sqrt{3}}{2}, \frac{1}{2}\rangle}(x^2+y^2)\Big|_{(0,1)} = \nabla(x^2+y^2)\Big|_{0,1} \cdot \left\langle \frac{\sqrt{3}}{2}, \frac{1}{2}\right\rangle$$
$$= \langle 2x, 2y\rangle\Big|_{0,1} \cdot \left\langle \frac{\sqrt{3}}{2}, \frac{1}{2}\right\rangle$$
$$= 1.$$

1.3. Finding Maxima and Minima. Find the point on the plane $z = 2x + y + 1$ which minimizes the distance to $(1, 0, 2)$. Check that this matches the result from using the distance formula from the plane.

1.4. **Gradient Vector.**



Estimate the gradient vector at $(-1, -3)$, then use this estimate to compute the directional derivative in the $\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle$ direction at that point.

Gradient vector direction $\approx \langle -1, 1 \rangle$

Normalized $\frac{\nabla f}{|\nabla f|} \approx \langle -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle$

Grad. vector length $\approx \frac{1}{\sqrt{2}}$

$\Rightarrow \quad \nabla f \approx \langle -\frac{1}{2}, \frac{1}{2} \rangle.$

$$D_{\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle} f = \nabla f \cdot \langle \tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}} \rangle = 0.$$