1. A detailed Solution

1.1. **Using the Chain Rule, I.** Suppose that f has a minimum at $(0,0)$. Let $\vec{r}(t) = \langle at, bt\rangle$ be a line traveling through the origin in the direction of $\langle a, b\rangle$. Show using the chain rule that

$$\frac{d^2}{dt^2} f(\vec{r}(t))\bigg|_{t=0} > 0$$

This is a chain rule problem. Letting

$$x(t) = at \quad y(t) = bt$$

we are asked to compute $\frac{d^2}{dt^2} f(x(t), y(t))$. Using the chain rule, we get

$$\begin{aligned}\frac{d^2}{dt^2} f(x(t), y(t) =& \frac{d}{dt}\left(\frac{d}{dt} f(x(t), y(t))\right)\\ =& \frac{d}{dt}(f_x x'(t) + f)y y'(t))\\ =& \frac{d}{dt}(af_x + bf_y)\end{aligned}$$

Key Insight: f_x and f_y are functions of $x(t), y(t)$, so we again have to use the chain rule.

$$\begin{aligned}=& a(f_{xx}x'(t) + f_{xy}y'(t)) + b(f_{yx}x'(t) + f_{yy}y'(t))\\ =& f_{xx}a^2 + 2f_{xy}ab + f_{yy}b^2\end{aligned}$$

Key Insight: Think of this expression as a polynomial of a. Then it is always greater than zero if the polynomial has no zeros, which occurs when the discriminant of the polynomial is negative. The discriminant of this polynomial (of a) is

$$(2f_{xy}b)^2 - 4(f_{yy}b^2)(f_{xx})$$

which simplifies to $-b^2(f_{xx}f_{yy} - (f_{xy})^2)$, which is the determinant used to find if f has a maximum, minimum or saddle point at a critical point. Since we know that we have a minimum, this gives us $f_{xx}f_{yy} - (f_{xy})^2 > 0$, so the polynomial $f_{xx}a^2 + 2f_{xy}ab + f_{yy}b^2$ has no root, and is always positive. It follows that

$$\frac{d^2}{dt^2} f(\vec{r}(t))\bigg|_{t=0} > 0$$