1. More On Derivatives

1.1. Using the Chain Rule, I. Let $s(t) = f(x(t), y(t))$, and assume that

$$\left.\frac{ds}{dt}\right|_{t=3} = 5$$

Additionally, suppose that you are told that

$$\left.\frac{\partial f}{\partial x}\right|_{(0,2)} = 1 \qquad \left.\frac{\partial f}{\partial y}\right|_{(0,2)} = 1$$

$$\left.\frac{dx}{dt}\right|_{t=3} = 3$$

$$y(3) = 0 \qquad x(3) = 2$$

What is $y'(3)$?

$$\frac{ds}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

at $t=3$

$$5 = \left.\frac{ds}{dt}\right|_3 = \left.\frac{\partial f}{\partial x}\right|_{(0,2)} \left.\frac{dx}{dt}\right|_{t=3} + \left.\frac{\partial f}{\partial y}\right|_{(0,2)} \left.\frac{dy}{dt}\right|_{t=3}$$

$$5 = 1 \cdot 3 + 1 \cdot \left.\frac{dy}{dt}\right|_{t=3}$$

$$\Rightarrow \left.\frac{dy}{dt}\right|_{t=3} = 2.$$

1.2. Chain Rule in Proofs. The function $f(x, y, z) = x^2 + y^2 + z^2$ measures the distance squared of a point from the origin. Show that if $\vec{r}(t) = \langle x(t), y(t), z(t)\rangle$ with $\vec{r}(0) = \langle 1, 0, 0\rangle$, and $\vec{r}(t) \cdot \vec{r}'(t) = 0$, that $f(x(t), y(t)) = 1$ for all t.
Conclude that the curve $\vec{r}(t)$ is contained in the unit sphere.

Want to show $f(\vec{r}(t)) = 1$

$$\Longleftarrow \quad f(\vec{r}(0)) = 1 \quad \text{and} \quad \frac{d}{dt} f(\vec{r}(t)) = 0.$$

$$\frac{d}{dt} f(\vec{r}(t)) = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}$$

$$= 2\langle x, y, z\rangle\Big|_{r(t)} \cdot \vec{r}'(t)$$

$$= 2\vec{r} \cdot \vec{r}'(t) = 0.$$

1.3. **Implicit Differentiation.** Let $f(x,y) = x^3 + y^2$. Suppose that we have a curve $\vec{r}(t) = \langle x(t), y(t) \rangle$ which traces out a level set of f so that

$$f(x(t), y(t)) = 2.$$

Additionally, suppose that

$$\langle x(0), y(0) \rangle = \langle 1, 1 \rangle.$$

Finally, assume that $x'(0) = 1$. Find the velocity vector $\vec{v}(0)$.

$\frac{df}{dt} = 0$ as $f(x(t), y(t))$ is constant

so $\frac{df}{dt} = f_x x' + f_y y'$

$= 3x\big|_{\langle 1,1\rangle} \cdot x'(0) + 2y\big|_{\langle 1,1\rangle} y'(0)$

$0 = 3 \cdot 1 + 2 \cdot y'(0) \Rightarrow y' = -3/2.$

Notice that the velocity vector of $\vec{r}(t)$ is tangent to the 2-level curve at the point $(1,1)$. Use this information to find a tangent line to level set at $(1,1)$.

$$\boxed{\ell(t) = \langle 1, -3/2 \rangle t + \langle 1, 1 \rangle.}$$