1. Derivatives of Multi-variable functions, I

1.1. Partial Derivatives+ Linear Approximation. Compute the partial derivatives of

$$f(x,y) = x^2 - y^2$$

at the point $(1,1)$. Find the tangent plane to f at the point $(1,1,0)$ and use this information to estimate the value of $f(1.5,1.5)$.

$$\frac{\partial f}{\partial x} = 2x$$

$$\frac{\partial f}{\partial y} = -2y$$

$$\vec{n}\Big|_{(x,y,f(x,y))} = \langle 2x, -2y, -1\rangle$$

Looking at $x=1$, $y=1$

$$\vec{n}\Big|_{(1.5,1.5,0)} = \langle 2, -2, -1\rangle$$

Eq^{n} plane is $\vec{n}\cdot(\vec{v} - p_0)$ & $p = (1.5, 1.5, 0)$

$\rightsquigarrow$ plane is $2x - 2y - z = 0 \Rightarrow z = 2x - 2y$

$$\Longrightarrow f(1.5, 1.5) \approx 2(1.5) - 2(1.5) = 0.$$

1.2. Limits. Show that

$$f(x,y) = \frac{2xy}{x^2+y^2}$$

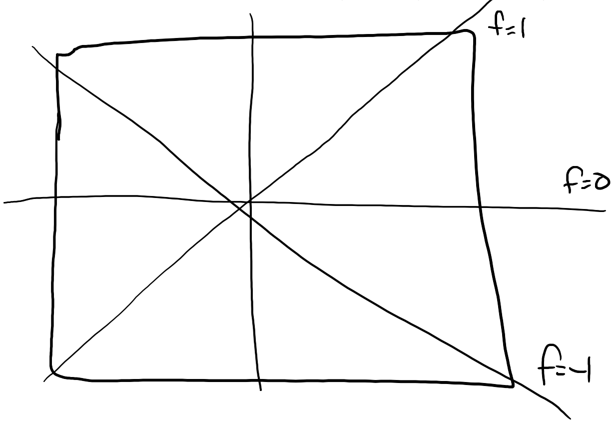
is not continuous by

- Finding the limit as t goes to zero along the lines $\langle t, t\rangle$ and $\langle t, -t\rangle$.

$$f(t,t) = \frac{2t^2}{t^2+t^2} = \frac{t^2}{t^2} \rightsquigarrow \lim_{t\to 0} \rightsquigarrow 1$$

$$f(t,-t) = \frac{-2t^2}{t^2+t^2} = -\frac{t^2}{t^2} \rightsquigarrow \lim_{t\to 0} \rightsquigarrow -1$$

- Making a contour plot with level sets $f=0$, $f=1$ and $f=-1$.



1.3. **Partial Derivatives + Linear Approximation II.** The total differential of f is given by

$$df = (2x + y)dx + (x)dy.$$

Suppose that $f(0, 1) = 1$. Give the equation of the tangent plane at $(0, 1, 1)$, and estimate the value of $f(1, 2)$.

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

$$\frac{\partial f}{\partial x} = 2x+y$$

$$\frac{\partial f}{\partial y} = x$$

$$\vec{n}_{(x,y,f(x,y))} = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, -1 \right\rangle$$

$$\vec{n}_{(0,1,1)} = \langle 2x+y, x, -1 \rangle \Big|_{\substack{x=0 \\ y=1}}$$

$$= \langle 1, 0, -1 \rangle$$

Eq$^{\text{n}}$ of plane is $\vec{n} \cdot (\vec{v} - p_0) = 0$ $\quad p_0 = (0, 1, 1)$

$$\langle 1, 0, -1 \rangle \cdot (\langle x, y, z \rangle - \langle 0, 1, 1 \rangle)$$

$$x - z + 1 = 0$$

Linear approximation $\Rightarrow f(x,y) \approx z = x+1$.

$$f(1,2) \approx 1+1 = 2 .$$