Worksheet, Sep 8

(1) A wooden stick is 10 meters long and 40 centimeters in diameter. A string wound around the stick in a spiral. The spiral winds around the stick 20 times (so the distance between each winding of the string is .5 meters.) How much string is used in the spiral?

Solution: Let's imagine that our stick goes "upward" in the z direction, and let's parameterize the string with a function $\vec{r}(t)$. Then the z-coordinate of $\vec{r}(t)$ should be continuously increasing, while the x and y coordinate should be tracing out a circle. So, we have that

$$\vec{r}(t) = \langle a\cos t, a\sin t, b\rangle$$

where a determines the radius, and b determines the rate at which this helix moves upward versus spiraling. The radius of the stick is $a = .4$, and the rate of winding is $b = \frac{.5}{2\pi}$.
This gives us

$$\vec{r}(t) = \langle \frac{2}{5}\cos t, \frac{2}{5}\sin t, \frac{t}{4\pi}\rangle$$

$$|\vec{r}(t)| = \sqrt{\left(\frac{-2}{5}\sin t\right)^2 + \left(\frac{2}{5}\cos t\right)^2 + \left(\frac{1}{4\pi}\right)^2} = \sqrt{\frac{8}{25} + \frac{1}{16\pi^2}}$$

The length is given by

$$\int_0^{40\pi} |\vec{r}(t)|dt = 40\pi\sqrt{\frac{8}{25} + \frac{1}{16\pi^2}}$$

(2) Let $\langle x(t), y(t)\rangle$ be a parametric curve. Let L be the length of the curve from $t = 0$ to $t = 1$. Make a geometric argument for why

$$L \geq |x(1) - x(0)|,$$

and prove this using the arc-length formula. When does this equality hold? (Draw pictures!)

Solution:A calculation shows

$$\begin{aligned} L &= \int_{t_0}^{t_1} \sqrt{(x')^2 + (y')^2}dt \\ &\geq \int_{t_0}^{t^1} \sqrt{(x')^2}dt \\ &\geq \int_{t^0}^{t_1} x'dt \\ &= x(1) - x(0) \end{aligned}$$

(3) For which values of t is the curve given by $\langle 1 - t^3, 1 - t^2, 1 + t\rangle$ parallel, skew, intersecting or tangent to the z axis?

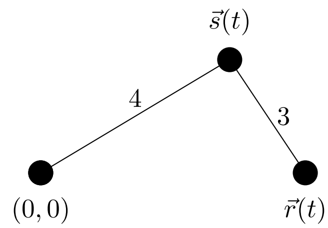
Solution: First, the curve $\langle 1 - t^3, 1 - t^2, 1 + t\rangle$ will intersect the z axis when both the x and y coordinate are zero; this only happens when $t = 1$. The curve is parallel to the z axis when it's velocity is parallel to the z axis: the velocity is

$$\langle 1 - 3t^2, 1 - 2t, 1\rangle$$

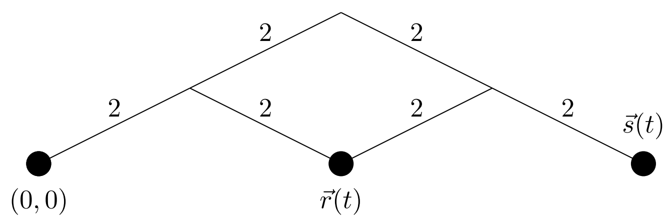
So, we need $1 - 3t^2 = 1 - 2t = 0$, which never occurs. So, the curve usually skew, except at $t = 1$.

A linkage is a contraption made from steel bars and hinges. The steel bars are not allowed to change length. A single hinge is fixed at the point $(0,0)$, but all the other hinges are allowed to rotate freely and move. A single hinge is called the "input point", while another hinge is called the "output point." By pushing the location of the input point around with a parameteric equation $\vec{r}(t)$, we get a new parametric equation $\vec{s}(t)$ at the output point. For each linkage, figure out the output parametric equation for the corresponding input parametric equation.

(1) Let $\vec{r}(t) = \langle 5\cos(t), 5\sin(t)\rangle$.



(2) Let $\vec{r}(t) = \langle x(t), y(t)\rangle$. (Note: the line going from the origin labeled 2 and 2 is always straight in this example.) This linkage is historically called a Pantograph.



(3) Let $\vec{r}(t) = \langle 1 + 2\cos\theta, \sin\theta\rangle$