Tetrahedra Notes

Let's look at this tetrahedra problem. First, some notation. I'm going to call the four corners of the tetrahedra $\vec{p}_0, \vec{p}_1, \vec{p}_2$ and $\vec{p}_3$. We'll label the edges of this tetrahedra with the vectors $\vec{e}_{12}$, which is suppose to be an edge from p_1 to p_2. Similarly, when I write $\vec{v}_{123}$, I'll mean the normal vector to the face containing vertices $\vec{p}_1, \vec{p}_2$, and $\vec{p}_3$ whose magnitude is the area of the face.
The problem asks us to show that

$$\vec{v}_{012} + \vec{v}_{123} + \vec{v}_{230} + \vec{v}_{301} = 0.$$

We can write each of the face vectors in term of the edge vectors by taking a cross product of edges perpendicular to that face. I'm going to take a very particular set of vectors to write out these edges:

$$\begin{aligned}\vec{v}_{012} =& \frac{1}{2}(\vec{e}_{21} \times \vec{e}_{20}) = \frac{1}{2}(\vec{e}_{12} \times \vec{e}_{02})\\ \vec{v}_{123} =& \frac{1}{2}(\vec{e}_{12} \times \vec{e}_{13})\\ \vec{v}_{230} =& \frac{1}{2}(\vec{e}_{03} \times \vec{e}_{02})\\ \vec{v}_{301} =& \frac{1}{2}(\vec{e}_{30} \times \vec{e}_{31}) = \frac{1}{2}(\vec{e}_{03} \times \vec{e}_{13})\end{aligned}$$

Using these representations of the normal vectors, our previous computation expands to

$$\begin{aligned}\vec{v}_{012} + \vec{v}_{123} + \vec{v}_{230} + \vec{v}_{301} =& \frac{1}{2}\left(\vec{e}_{12} \times \vec{e}_{02} + \vec{e}_{12} \times \vec{e}_{13} + \vec{e}_{03} \times \vec{e}_{02} + \vec{e}_{03} \times \vec{e}_{13}\right)\\ =& \frac{1}{2}\left((\vec{e}_{12} + \vec{e}_{03}) \times (\vec{e}_{02} + \vec{e}_{13})\right)\end{aligned}$$

We've now reduced this to showing that $(\vec{e}_{12} + \vec{e}_{03})$ and $(\vec{e}_{02} + \vec{e}_{13})$ are parallel. This follows as

$$\begin{aligned}(\vec{e}_{12} + \vec{e}_{03}) = \vec{p}_2 - \vec{p}_1 + \vec{p}_3 - \vec{p}_0\\ (\vec{e}_{02} + \vec{e}_{13}) = \vec{p}_0 - \vec{p}_1 + \vec{p}_3 - \vec{p}_1.\end{aligned}$$