Discussion Worksheet, Aug 28

0.1. **Computing a Cross Product.** Find a vector which is mutually orthogonal to $\langle 1,0,1\rangle$ and $\langle 0,1,2\rangle$.

Solution: This is taking a cross product of the two vectors

$$\begin{aligned}\langle 1,0,1\rangle \cdot \langle 0,1,2\rangle =& \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 1 \\ 0 & 1 & 2 \end{vmatrix} \\ =& \begin{vmatrix} 0 & 1 \\ 1 & 2 \end{vmatrix} \hat{i} - \begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} \hat{j} + \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \hat{k} \\ =& -\hat{i} - 2\hat{j} + \hat{k} \\ =& \langle -1,-2,1\rangle .\end{aligned}$$

A quick check with the dot product confirms that this is mutually orthogonal to the two desired vectors.

0.2. **Volume of a parallelepiped.** Find the volume of the parallelepiped whose edges are given by the vectors $\langle 0,1,2\rangle$ and $\langle -1,-2,1\rangle$, $\langle 1,0,1\rangle$

Solution: We can use the triple product formula to get the volume of a parallelepiped, by taking

$$\langle -1,1,1\rangle \cdot (\langle 1,0,1\rangle \cdot \langle 0,1,2\rangle) .$$

We've previously computed the cross product on the right, so this simplifies to

$$\langle -1,1,1\rangle \cdot \langle 0,1,2\rangle$$

which is $-1 \cdot (0) + (-2) \cdot 1 + 1 \cdot 2 = 0$.

0.3. **Triple Products.** Show (by computation) that

$$\vec{u} \cdot (\vec{v} \times \vec{w}) = \vec{v} \cdot (\vec{w} \times \vec{u}),$$

and explain why this geometrically makes sense.

Solution: Both sides are computing the volume of the same parallelepiped, so they should have the same solution. We can check this by writing out in components

$$\vec{u} = \langle u_x, u_y, u_z \rangle$$
$$\vec{v} = \langle v_x, v_y, v_z \rangle$$
$$\vec{w} = \langle w_x, w_y, w_z \rangle$$

then computing

$$\vec{u} \cdot (\vec{v} \times \vec{w}) = \langle u_x, u_y, u_z \rangle \cdot (\langle v_x, v_y, v_z \rangle \times \langle w_x, w_y, w_z \rangle)$$

and checking that it is equal to

$$\vec{v} \cdot (\vec{w} \times \vec{u}) = (\langle v_x, v_y, v_z \rangle \cdot (\langle w_x, w_y, w_z \rangle \times \langle u_x, u_y, u_z \rangle)$$

0.4. **Concept Check.** Why is $|\vec{u} \cdot (\vec{v} \times \vec{w})| \leq |\vec{u}||\vec{v}||\vec{w}|$?

Solution: Think about how we can express the magnitude of the dot product or cross product by using $\sin\theta$ and $\cos\theta$, and about how large these functions can be.