QUIZ, AUGUST 28

NAME:

0.1. **Lengths.** Find a unit vector pointing in the direction of $\langle 1,1,2\rangle$.

Solution: Let $\vec{v} = \langle 1,1,2\rangle$. To make this unit vector, we scale it by the reciprocal of its length. The length of $\vec{v}$ is

$$|\vec{v}| = \sqrt{1^2+1^2+2^2} = \sqrt{6}$$

so our unit vector would be

$$\frac{\vec{v}}{|\vec{v}|} = \left\langle \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}} \right\rangle.$$

0.2. **Dot Product.** Find a unit vector $\vec{u}$ whose component onto the vector $\vec{v} = \langle 1,0\rangle$ is

$$\text{comp}_{\vec{v}}(\vec{u}) = \frac{1}{2}.$$

Solution: Let's have $\vec{u} = \langle x,y\rangle$. Then the component onto $\langle 1,0\rangle$ is

$$\begin{aligned}\text{comp}_{\vec{v}}(\vec{u}) =& \frac{\vec{v}\cdot\vec{u}}{|\vec{v}|}\\ =& \frac{\langle 1,0\rangle\cdot\langle x,y\rangle}{|\langle 1,0\rangle|}\\ =& x\end{aligned}$$

So we get $x = \frac{1}{2}$. We then need to find a vector which has x-component $\frac{1}{2}$, and length 1.

$$\sqrt{x^2+y^2} = \sqrt{1/4+y^2} = 1$$

from which we conclude that $y = \pm\frac{\sqrt{3}}{2}$.

0.3. **Some Geometry.** Suppose that $\vec{u}, \vec{v}$ and $\vec{w}$ are vectors corresponding to the edges of an equiliateral triangle, so that

$$|\vec{u}| = |\vec{v}| = |\vec{w}|.$$

Show that the angle between $\vec{v}$ and $\vec{u}$ is $\frac{\pi}{3}$ radians.
Solution: Since $\vec{v}, \vec{w}$ and $\vec{u}$ make a triangle, we can write

$$\vec{v} = \vec{u} - \vec{w}.$$

Using the fact that this is a equilateral triangle,

$$\begin{aligned} |\vec{v}|^2 =& (\vec{u} + -\vec{w}) \cdot (\vec{u} - \vec{w}) \\ =& |\vec{u}|^2 + |\vec{w}|^2 - 2\vec{u} \cdot \vec{w} \\ =& 2|\vec{v}|^2 - 2\vec{u} \cdot \vec{w} \end{aligned}$$

From this we conclude that

$$\vec{u} \cdot \vec{w} = \frac{|\vec{v}|^2}{2} = \frac{|\vec{u}||\vec{w}|}{2}$$

Applying the law of cosines, we get $\cos\theta = \frac{\vec{u} \cdot \vec{w}}{|\vec{u}||\vec{w}|} = \frac{1}{2}$, and conclude that $\theta = \frac{\pi}{3}$.

0.4. **Bonus Problem, worth no points.** This problem is taken from *The Curious Incident of the Dog in the Night-time,* and can be proven using the vector geometry we've developed in class.

> Prove the following result: A triangle with sides that can be written in the form $n^2 + 1$, $n^2 - 1$ and $2n$ (where $n > 1$) is right-angled.
> Show, by means of a counterexample, that the converse is false.