Discussion Worksheet, Aug 25

0.1. **Right angles and Dot Products.** Suppose that $\vec{v}, \vec{u}$, and $\vec{w}$ make three edges of a triangle. Suppose we have the identity

$$|\vec{v}|^2 = |\vec{u}|^2 + |\vec{w}|^2.$$

Show that if this identity holds, then $\vec{u} \cdot \vec{w} = 0$, and the triangle must be right.

Solution: Since these three vectors form a triangle, we have the identity

$$\vec{v} = \vec{u} + \vec{w}$$

and we may compute

$$\begin{aligned} |\vec{v}|^2 =& \vec{v} \cdot \vec{v} \\ =& (\vec{u} + \vec{w}) \cdot (\vec{u} + \vec{w}) \\ =& \vec{u} \cdot \vec{u} + 2\vec{u} \cdot \vec{w} + \vec{w} \cdot \vec{w} \\ =& |\vec{u}|^2 + 2\vec{u} \cdot \vec{w} + |\vec{w}|^2 \end{aligned}$$

Since we've assumed that $|\vec{v}|^2 = |\vec{u}|^2 + |\vec{w}|^2$, we can conclude that $2\vec{u} \cdot \vec{w} = 0$.

0.2. **Component of vectors.** Find a unit vector that points in the same direction as

$$\langle 1, 3, 1 \rangle$$

Find the component of the vector $\langle 2, 1, 1 \rangle$ onto this vector.

Solution: For the first part, we know that the unit vector that points in the $\vec{v}$ direction is $\vec{v}/|\vec{v}|$. In this case we have the vector

$$\frac{1}{\sqrt{1^2 + 3^2 + 1^2}} \langle 1, 3, 1, \rangle = \langle \frac{1}{\sqrt{11}}, \frac{3}{\sqrt{11}}, \frac{1}{\sqrt{11}} \rangle$$

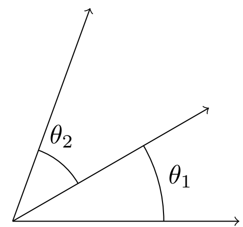
The component of $\vec{u}$ onto $\vec{v}$ is given by

$$\text{Comp}_{\vec{v}} \vec{u} = \frac{\vec{v} \cdot \vec{u}}{|\vec{v}|} = \frac{2 \cdot 1 + 1 \cdot 3 + 1 \cdot 1}{\sqrt{11}} = \frac{6}{\sqrt{11}}$$

0.3. **Angles II (Hard!)** Prove the identity

$$\cos(\theta_1 + \theta_2) = \cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2)$$

by using the dot product formula for angles and the following setup of vectors.



Solution: We use both the formula for sin and cosine from determinant and dot product respectively. Let's call the vectors (from bottom to top) $\vec{a}, \vec{b}, \vec{c}$, and assume that they are unit length. Then we have

$$\begin{aligned}
\sin\theta_1 &= \det(\vec{a}, \vec{b}) \\
\sin\theta_2 &= \det(\vec{b}, \vec{c}) \\
\cos\theta_1 &= \vec{a} \cdot \vec{b} \\
\cos\theta_2 &= \vec{b} \cdot \vec{c} \\
\cos(\theta_1 + \theta_2) &= \vec{a} \cdot \vec{c}
\end{aligned}$$

Then we compute

$$\begin{aligned}
(\vec{a} \cdot \vec{b})(\vec{b} \cdot \vec{c}) - \det(\vec{a}, \vec{b})\det(\vec{b}, \vec{c}) &= (a_x b_x + a_y b_y)(b_x c_x + b_y c_y) - (a_x b_y - a_y b_x)(b_x c_y - b_y c_x) \\
&= (a_x b_x b_y c_y + a_y b_y b_x c_x + a_x b_x b_x c_x + a_y b_y b_y c_y) \\
&\quad - (a_x b_x b_y c_y - a_x b_y b_y c_x - a_y b_x b_x c_y + a_y b_x b_y c_x)
\end{aligned}$$

Some terms cancel

$$\begin{aligned}
&= (a_x b_x b_x c_x + a_y b_y b_y c_y + a_x b_y b_y c_x + a_y b_x b_x c_y) \\
&= a_x c_x (b_x b_x + b_y b_y) + a_y c_y (b_x b_x + b_y b_y) \\
&= (a_x c_x + a_y c_y)|\vec{b}| \\
&= \vec{a} \cdot \vec{c} \\
&= \cos(\theta_1 + \theta_2)
\end{aligned}$$