1. Let $\vec{F}$ be vector field on a region $U \subset \mathbb{R}^2$.

   (a) Prove the 2-dimensional divergence theorem:

   $$\int_U \nabla \cdot \vec{F} dA = \int_C F \cdot \vec{n}\, ds.$$

   Here, $\vec{n}$ is the normal vector to the curve C, and C bounds the region U.
   Solution: The idea is to convert this to Green's theorem. Let $\vec{F} = \langle P, Q \rangle$. Let $\vec{F} = \langle -Q, P \rangle$.

   $$\begin{aligned}\int_U \nabla \cdot \vec{F} dA &= \int_U P_x + Q_y \\ &= \int_U P_x - (-Q)_y dA \\ &= \int_C \vec{G} d\vec{r} \\ &= \int_C \vec{F}\vec{n}\, ds\end{aligned}$$

   (b) Let $\vec{F}$ be the vector field

   $$\left\langle \frac{x}{x^2+y^2}, \frac{y}{x^2+y^2} \right\rangle$$

   Compute $\int_C \vec{F} \cdot \vec{n}\, ds$ for any curve C which does not go around the origin.
   Solution: If we compute out the divergence we get:

   $$\frac{x^2+y^2-x(2x)}{(x^2+y^2)^2} + \frac{x^2+y^2-y(2y)}{(x^2+y^2)^2} = 0$$

   whenever you do not contain the origin. So by the 2-dimensional divergence theorem, integrating around any curve that does not contain the origin gives :

   $$\int_C \vec{F} \cdot \vec{n}\, ds = \int U \mathrm{div} F dA = 0$$

   (c) Compute $\int_C \vec{F} \cdot \vec{n}\, ds$ for any curve C which does go around the origin.
   Solution: This doesn't end up working you are integrating around the singularity. However, this should be independent of the curve that goes around the origin, so let's pick a nice one, say that unit circle. On the unit circle, $\vec{n} = \vec{F}$. Then

   $$\int_C \vec{F} \cdot \vec{n} ds = \int_C 1 ds = 2\pi.$$

2. (a) Let $\vec{F}$ be the vector field $\langle 3x, 2y, 0\rangle$. Integrate

$$\iint_S \vec{F} \cdot dS,$$

where S is the sphere of radius 3 centered at the origin.

(b) Let $\vec{F}$ be the vector fields $\langle -3y, 3x, 0\rangle$. Integrate

$$\iint_H \vec{F} \cdot dH,$$

where H is the upper half hemisphere of the sphere of radius 3 centered at the origin.

(c) Let $\vec{F} = \langle xze^y, -xze^{-y}, z\rangle$. Integrate this vector field over the region

$$x + y + z = 1$$

in the first octant, oriented downward.

3. Define the *Laplacian* of a function $f : \mathbb{R}^3 \to \mathbb{R}$ to be the quantity

$$\nabla^2 f = \operatorname{div} \operatorname{grad} f.$$

Suppose that $\nabla^2 f = 0$.

(a) Show that $f(\rho, \theta, \phi) = \frac{1}{\rho}$ has the property that $\nabla^2 f = 0$.

(b) Suppose that p is a maximum on for f on some region U i.e. $f(x) \leq f(p)$ for every point $x \in U \subset \mathbb{R}^3$. Make an argument for why $p \in \partial U$, the boundary of U.

(c) Conclude that a function with $\nabla^2 f = 0$ has no absolute maxima or minima on $\mathbb{R}^3$.

Solution: For the first part, convert back to Cartisian coordinates. Then

$$\begin{aligned}\nabla^2 f =& \nabla \cdot \nabla\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right) \\ =& \nabla \cdot ((x^2+y^2+z^2)^{-3/2}\langle -x, -y, -z\rangle \\ =& 0\end{aligned}$$

The idea for the second part is really nice. Suppose for contradiction that p was the maximum of a function. Then the gradient vector field around p points towards p, because p is a maxima for the function. This means that $\operatorname{grad} f$ should be negative near p. But $\operatorname{grad} f$ is assumed to be 0 everywhere.

4. (a) Sketch the surface parameterized by the equations

$$\begin{aligned} x(s,t) =& (\sin(t)+2)\cos s \\ y(s,t) =& (\sin(t)+2)\sin s \\ z(s,t) =& \cos(t) \end{aligned}$$

(b) Find the volume of the figure.

Solution: This draws out a torus. It's easiest to compute this integral instead in cylinderical coordinates, where the region is given by

$$\begin{gathered} 0 \leq \theta \leq 2\pi \\ 1 \leq r \leq 2 \\ -\sqrt{1-(r-1)^2} \leq z \leq \sqrt{1-(r-1)^2} \end{gathered}$$

5. Let U_{ab} be the box

$$-a \leq x \leq a \qquad\qquad -b \leq y \leq b \qquad\qquad 0 \leq z \leq \frac{1}{ab}.$$

Let S_{ab} be the portion of the boundary of U_{ab} with $z > 0$. Find values for a and b that maximize the quantity

$$\int_{S_{ab}} \left(\left(-\frac{1}{x^2} - \frac{1}{y^2} \right) \vec{k} \right) \cdot dS$$

up to the constraint that $\operatorname{vol} U_{ab} = 1$.