1. Let $\vec{F}$ be vector field on a region $U \subset \mathbb{R}^2$.

   (a) Prove the 2-dimensional divergence theorem:

   $$\int_U \nabla \cdot \vec{F} dA = \int_C F \cdot \vec{n}\, ds.$$

   Here, $\vec{n}$ is the normal vector to the curve C, and C bounds the region U.

   (b) Let $\vec{F}$ be the vector field

   $$\left\langle \frac{x}{x^2+y^2}, \frac{y}{x^2+y^2} \right\rangle$$

   Compute $\int_C \vec{F} \cdot \vec{n}\, ds$ for any curve C which does not go around the origin.

   (c) Compute $\int_C \vec{F} \cdot \vec{n}\, ds$ for any curve C which does go around the origin.

2. (a) Let $\vec{F}$ be the vector field $\langle 3x, 2y, 0 \rangle$. Integrate

   $$\iint_S \vec{F} \cdot dS,$$

   where S is the sphere of radius 3 centered at the origin.

   (b) Let $\vec{F}$ be the vector fields $\langle -3y, 3x, 0 \rangle$. Integrate

   $$\iint_H \vec{F} \cdot dH,$$

   where H is the upper half hemisphere of the sphere of radius 3 centered at the origin.

   (c) Let $\vec{F} = \langle xze^y, -xze^{-y}, z \rangle$. Integrate this vector field over the region

   $$x + y + z = 1$$

   in the first octant, oriented downward.

3. Define the *Laplacian* of a function $f : \mathbb{R}^3 \to \mathbb{R}$ to be the quantity

   $$\nabla^2 f = \operatorname{div} \operatorname{grad} f.$$

   Suppose that $\nabla^2 f = 0$.

   (a) Show that $f(\rho, \theta, \phi) = \frac{1}{\rho}$ has the property that $\nabla^2 f = 0$.

   (b) Suppose that p is a maximum on for f on some region U i.e. $f(x) \leq f(p)$ for every point $x \in U \subset \mathbb{R}^3$. Make an argument for why $p \in \partial U$, the boundary of U.

   (c) Conclude that a function with $\nabla^2 f = 0$ has no absolute maxima or minima on $\mathbb{R}^3$.

4. (a) Sketch the surface parameterized by the equations

   $$\begin{aligned} x(s,t) =& (\sin(t) + 2) \cos s \\ y(s,t) =& (\sin(t) + 2) \sin s \\ z(s,t) =& \cos(t) \end{aligned}$$

   (b) Find the volume of the figure.

5. Let U_{ab} be the box

   $$-a \leq x \leq a \qquad -b \leq y \leq b \qquad 0 \leq z \leq \frac{1}{ab}.$$

   Let S_{ab} be the portion of the boundary of U_{ab} with $z > 0$. Find values for a and b that maximize the quantity

   $$\int_{S_{ab}} \left(\left(-\frac{1}{x^2} - \frac{1}{y^2} \right) \vec{k} \right) \cdot dS$$

   up to the constraint that $\operatorname{vol} U_{ab} = 1$.