1

There is a prison run by a strange and mathematically inclined warden. The Warden assembles all 100 of his prisoners in a room, and gives them the following puzzle:
"Every day, a Guard will bring one of you (chosen at random) into a room. There will be a lightbulb in the room, and it will be on or off. I will give you the option of turning the lightbulb on or off. The Guard will then ask you if all your fellow prisoners have been in the room yet.

- If you stay silent, things will go on as normal, and tomorrow another random prisoner will be brought into the room.
- If you say yes, and you are incorrect, I will sentence you all to a life in prison without parole.
- If you say yes, and are correct, then you will all go free."

"Now, I promise you that I will not change the state of lightbulb. Furthermore, to be fair to you I will now give you a few minutes to discuss a strategy before putting you all back in your cells, where you will remain in total isolation until you solve the puzzle."
How can the prisoners best insure that there incarceration only lasts for a finite time?

* * *

One Prisoner is designated the special prisoner. His job is to only put out the light. Every other prisoner is instructed to only turn on the light one time. When the special prisoner has turned off the light 100 times, they can say that every other prisoner has been in the room.

2

There is a prison run by a strange and mathematically inclined warden. The Warden assembles all 100 of his prisoners in a line. He then puts a red hat or a blue hat on each of the 100 prisoners.
No prisoner can see the color of the hat on his or her head, but can see the hats on all of the prisoners in front of them in the line. The Warden starts at the end of the line, and barks, "What color is your hat?!", to which the prisoner is only allowed to reply "Red!" or "Blue!"
The other prisoners can hear what the prisoners behind them say.
If a prisoner correctly identifies the color of their own hat, then they are allowed to go free, otherwise, they are condemned to a lifetime of doing pushups.

Can you come up with a strategy for the prisoners that saves

- At least 1 prisoner?

- At least 50 prisoners?
- At least 51 prisoners?
- Almost all of the prisoners?

* * *

Think even/odd. The first prisoner looks at all of the hats in front of them. If there are an even number of red hats, they say blue!. By knowing the parity of the hats in front of them, the second prisoner can figure out what the color of their own hat is. By knowing the color of the hats behind him, the colors of hats in front of him, and the parity, a prisoner can figure out the color of their own hat.

3

There is a prison run by a strange and mathematically inclined warden. The warden is throwing a party for his guests, and has 1000 barrels of delicious fruit juice prepared for his guests. Unfortunately, during the middle of the night, one of the prisoners poisoned a barrel of juice.
The warden tells the prisoners that they must identify which barrels are poisoned by tomorrow morning, or he will punish them all severely. Unfortunately, the prisoners no longer remember which barrel is poisoned. They know that the poison takes 12 hours to take effect, so they can test the juice on the prisoners in order to figure out which barrels are poisoned, but they only have time to run such a test once. For example, you could take 1000 prisoners, and have them each sip from a different barrel; the next morning, by checking which prisoners are sick, the warden will be able to identify which barrels were poisoned. What is the minimal number of prisoners required to find the poisoned bottle.

* * *

Think Binary. Label the bottles in binary code. The first prisoner drinks from bottles labeled with a 1 in the 1's place, the second prisoner drinks from bottles labeled with a 1 in the 2's place, the third prisoner drinks from bottles labeled with a 1 in the 4's place, and so on. The sick prisoners will then "read out" the poisoned bottle in binary.

4

There is a prison run by a strange and mathematically inclined warden. He gathers 20 prisoners in a room, and gives them colorful hats. No prisoner can see their own hat, but can see all the other hats. The warden then says "At least one of you is wearing a red hat. I will ask all of you together to raise your

hand if you are wearing a red hat. If one of you raises your hand incorrectly, I will take away all of your meals for the week. If everybody who is wearing a red hat correctly identifies themselves, I will let you all go free. There is no penalty for not raising your hand. Also, I will continue to ask the group 'Are you wearing a red hat' until either somebody is wrong, or all the red hats have been identified."
So he begins: "Are you wearing a red hat? "
As nobody knows the color of their own hat, nobody raises their hands. And he asks the group again, "Are you wearing a red hat? " Is there a way for a group of rational prisoners to save themselves ?

* * *

Suppose there was only 1 red hat. Then the person wearing the hat knows that they have it, so they mention it on the first round.
Suppose there were 2 red hats. Then each person wearing a red hat sees another person wearing a red hat, so they say nothing on the first round. Now, seeing that nobody said anything on the first round, they can each infer that there is a second red hat; which means that they are wearing it. So on the second round, they say that they are wearing a red hat.
The rational prisoner therefore claims that they have a red hat on the nth round if they see $n-1$ red hats.

5

There is a pirate ship commanded by a just, communal, and mathematically minded pirate captain. Though she is a fearsome pirate, she lives by the pirate code when distributing booty to her mates. The pirate code states:

> The Pirate captain shall come up with a plan to distribute booty. That plan then goes to the crew for a vote. If the vote is in favor or a tie, the plan passes and booty is distributed. The pirates then go off to plunder more treasure. If the vote fails to pass, then the captain shall be forced to walk the plank and jump into shark infested waters. The first mate shall become the captain, the second mate shall be promoted to first mate, and so on (the $(n+1)$-mate will be promoted to n-mate.) The new captain will come up with a plan, and the process above repeated until booty has been distributed.

The pirate captain knows her 50 crew members well. In particular, she knows that all pirates vote on the following preferences:

- Pirates look out for their own neck before thinking about plunder— that is, no pirate will propose a plan to distribute booty that will cause them to walk the plank if they can afford to.
- Pirates prefer more treasure to less treasure, and don't really care about friendships. They don't make promises beyond following the pirate code.

One day, the pirates come across 100 gold coins to distribute. What is the final distribution of gold coins among the pirates?

* * *

It turns out if there is an odd number of pirates, then only the even number pirates need to be minimally paid, and if there is an even number of pirates, only the odd pirates need to be minimally paid. Think about the situation for a few pirates:

- 1 pirate gets all the money.
- 2 pirates, the first mate gets all of the money.
- 3 pirates, the first mate gets 1 coin, and captain gets everything else.
- 4 pirates, the first and 3rd mate get 1 coin, and the captain gets everything else.

See if you can make this argument rigorous using mathematical induction.

7

Jane and Fred have died at 11PM (and gone to heaven and hell respectively.) In heaven, god makes the following deal:

- At 11PM, Jane will receive 1 dollar. Then god will take this dollar away from her.
- At 11:30PM, Jane will receive 2 dollars. This gives her a little stack of money. God then removes the top dollar from this stack (leaving her with one dollar leftover.)
- At 11:45, Jane will receive 3 dollars on top of her stack. God then removes the top dollar from this stack (leaving her with 3 dollars.)
- At 11:52:30, Jane will receive 4 dollars on top of her stack. God then removes the top dollar from this stack (leaving her with 6 dollars.)

and so on... Meanwhile, in hell, Fred gets the following deal:

- At 11PM, Fred will receive 1 dollar. Then the devil will take this dollar away from him.
- At 11:30PM, Fred will receive 2 dollars. This gives him a little stack of money. The devil then removes the bottom dollar from this stack (leaving him with one dollar leftover.)
- At 11:45, Fred will receive 3 dollars on top of his stack. The devil then removes the bottom dollar from this stack (leaving him with 3 dollars)

- At 11:52:30, Fred will receive 4 dollars on top of his stack. The devil then removes the bottom dollar from this stack (leaving him with 6 dollars.)

and so on. How much money do Jane and Fred have at midnight?

* * *

This is an example of limits not commuting. Jane keeps all of her money, and Fred has nothing. Imagine every dollar that Fred receives being labeled in the order that he receives them. At midnight, suppose for contradiction that Fred still has some money left over. In particular, he has a dollar bill labeled n left over. However, the devil removes this dollar bill on the nth step. This is a contradiction! The difference between Jane's money at midnight and Fred's money at midnight is analogous to the difference between these two sums:

$$\sum_{n=1}^{\infty}(n-1)$$

$$\left(\sum_{n=1}^{\infty} n\right) - \left(\sum_{n=1}^{\infty} 1\right)$$

8

Alice and Bob are in a long distance relationship, and are about to get married. This means that Bob needs to send Alice a ring. Alice has an envious neighbor Eve who has been trying to put an end to Alice and Bob's happy relationship for years. Every package that travels between Bob and Alice is searched by Eve, unless it is shipped inside a padlocked box. Bob has a lot of padlocks for this purpose; however, only he has the keys to his padlocks. Similarly, Alice has many padlocks so that she can ship things securely to Bob; however, only she has the keys to her own padlocks. How can Bob get a wedding ring to Alice?

* * *

First, Bob sends a box with the ring in it, and the box is padlocked. Then Jane applies one of her padlocks to the box, and sends the box (now with 2 padlocks on it) back to Bob. Bob then removes his padlock, and sends the box back to Jane. Jane gets a ring, and they live happily ever after.

9

8 math graduate students and their advisors are at a party. Many handshakes took place, but no one shook the hand of their own advisor. I asked everybody how many hands they had shaken, and everybody gave a different answer. How many hands did my advisor shake?

* * *

Since there are 15 people besides myself, and the number of possible handshakes vary between 0 and 14, this means that every value between 0 and 14 hands were shook. It is clear that the person who shakes n hands is the student of the professor who shakes $14 - n$ hands. This leaves 1 leftover (the person who has shook 7 hands.) This person is my advisor.

10

A picture is being hung by a wire and 100 of nails, like drawn below. Can you come up with a way of hanging the picture by a wire and a 100 nails so that if any one nail is removed, the picture falls?

$* * *$

This picture is hard to draw, so look up Borromean Rings for inspiration, and think about taking Math 191 next semester! There is a cool proof of this using fundamental groups of knots.