Quiz 10

Name:

Problem 1: True False. Mark true or false. If false, provide a counterexample or reason.

(a) For every vector field $\vec{F}$ on $\mathbb{R}^3$, $\text{curl}(\text{div}\,\vec{F}) = 0$
Solution: False. We have that $\text{div}\,(\text{curl}\,\vec{F}) = 0$.

(b) For every function $f : \mathbb{R}^3 \to \mathbb{R}$, $\text{div}\,(\text{grad}\, f) = 0$
Solution: False. We have that $\text{curl}\,\text{grad}(F) = 0$.

(c) For every function $f : \mathbb{R}^3 \to \mathbb{R}$, and every closed loop γ, we have that

$$\oint_\gamma \text{grad}\, f = 0.$$

Solution: True. This is the fundamental theorem of line integrals.

(d) Suppose $\text{div}\,\vec{F} = 0$ on a simply connected region U. Then on U, $\vec{F} = \text{grad}\, f$ for some $f : \mathbb{R}^3 \to \mathbb{R}$.
Solution: False. If $\text{div}\,\vec{F} = 0$ on a simply connected region U, then $\vec{F} = \text{curl}\,\vec{G}$ for some vector field $\vec{G}$

Problem 2: Integral computation. Compute the line integral over the curve C implicitly given by $x^2 + y^2 = 4$:

$$\oint_C x^2 y\; dx - xy^2\; dy$$

Solution: We use Green's theorem. Then if $P = x^2 y, Q = xy^2$, the quantity $Q_x - P_y = -(x^2 + y^2)$. Converting to polar coordinates, $-(x^2 + y^2) = r^2$. Doing the integral out in polar yields

$$\int_0^{2\pi} \int_0^2 (r^2) r\; dr d\theta = \int_0^{2\pi} 4\; d\theta = 8\pi$$

Problem 3: Curl and Leibniz. Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a differentiable function, and if $\vec{F} = \langle P, Q, R\rangle$ be a vector field. Define $f \cdot \vec{F}$ to be the vector field $\langle f \cdot P, f \cdot Q, f \cdot R\rangle$. Prove the product rule:

$$\operatorname{curl}(f \cdot \vec{F}) = (\operatorname{grad} f) \times \vec{F} + f \cdot \operatorname{curl} F$$

Solution:

$$\begin{aligned}
\operatorname{curl}(f \cdot \vec{F}) =& \operatorname{curl}\langle fP, fQ, fR\rangle \\
=& \langle (fQ)_z - (fR)_y, -(fP)_z + (fR)_x, (fP)_y - (fQ)_x\rangle \\
=& \langle f_zQ + fQ_z - f_yR - fR_y, -f_zP - fP_z + f_xR + fR_x, f_yP + fP_y - f_xQ - fQ_x\rangle \\
=& \langle f_zQ - f_yR, -f_zP + f_xR, f_yP - f_xQ\rangle + \langle fQ_z - fR_y, -fP_z + fR_x, fP_y - fQ_x\rangle \\
=& \langle f_x, f_y, f_z\rangle \times \langle P, Q, R\rangle + f\langle Q_z - fR_y, -P_z + fR_x, P_y - fQ_x\rangle \\
=& (\operatorname{grad} f) \times \vec{F} + f \cdot \operatorname{curl} \vec{F}
\end{aligned}$$

Interesting Puzzle, will not be graded. A picture is being hung by a wire and a pair of nails, like drawn below. Can you come up with a way of hanging the picture by a wire and a wire and a pair of nails so that if either nail is removed, the picture falls?