1. Find all of the points where the function $f(x,y,z) = x^2 + y^2 + z^2$ achieves a maximal and minimal value over unit cone

$$\{(x,y,z) \mid 0 \le z\ , z \le 1 - \sqrt{x^2+y^2}\}.$$

Solution: We use Lagrange multipliers to solve this.

- Check for Critical points on the interior of the region. This is solving $\nabla f = 0$ on the interior region.

$$\nabla f = \langle 2x, 2y, 2z \rangle$$

This is equal to 0 only at the origin.

- Now check for critical points on the boundary given by $0 = g(x,y,z) = 1 - z - \sqrt{x^2+y^2}$. Notice that Lagrange multipliers will not work when $x = y = 0$, because g has a singularity here. We need to solve $\lambda \nabla g = \nabla f$, which is solving

$$\lambda \left\langle \frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}, -1 \right\rangle = \langle 2x, 2y, 2z \rangle$$

Rescaling $\lambda = 2\mu\sqrt{x^2+y^2}$, we are solving

$$\mu \left\langle 2x, 2y, \sqrt{x^2+y^2} \right\rangle = \langle 2x, 2y, 2z \rangle$$

This tells us that either $\mu = 1$, or $x = y = 0$.

  - In the case where $\mu = 1$, we have that $z = \sqrt{x^2+y^2}$. This tells us that $z = 1 - z$, or that $z = \frac{1}{2}$.
  - The case $x = y = 0$, is not possible, as g has a singularity here.

This gives us 2 possible critical points. $z = 1/2$ has a value of $f(x, y, \frac{1}{2}) = 3/4$, and $f(0,0,1) = 1$.

- On the lower boundary, we are trying to maximize over the constraint $z = 0$. This is the same as solving

$$\lambda \langle 0, 0, 1 \rangle = \langle 2x, 2y, 2z \rangle$$

This constraint tells us that $\lambda = 0$, $x = y = z = 0$. However, this is already a critical point that we got for solving on the interior.

- Finally, we need to check at the intersection of the two constraints. This is solving

$$\lambda \left\langle \frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}, -1 \right\rangle + \mu \langle 0, 0, 1 \rangle = \langle 2x, 2y, 2z \rangle$$

When $\lambda = 2\sqrt{x^2+y^2}$, and $\mu = z + \sqrt{x^2+y^2}$, this is always satisfied by all x, y, z. Therefore, a maxima occurs when $x^2 + y^2 = 1$, which are all the possible solutions to both constraints.

Wrapping everything together, we have 4 kinds of critical values:

- $x = y = z = 0$. Here $f(0,0,0) = 0$. This is the absolute minima.
- $x = y = 0$, $z = 1$. Here $f(0,0,1) = 1$. This corresponds to a singularity on the cone, and it is a maximia.
- $x^2 + y^2 = 1$, $z = 0$. Here f takes a value of 1. This corresponds to critical points on both constraints.
- $z = \frac{1}{2}$, $x^2 + y^2 = \frac{1}{2}$. This corresponds to a minimal value on the "hat" portion of the cone's surface, but is not an absolute max or min of f.

2. Find the cone with largest volume that has surface area 1.

Solution: We can solve this by maximizing the volume of a cone

$$V(r,h) = \frac{1}{3}\pi r^2 h$$

to the constraint that

$$SA(r,h) = \pi r(r + \sqrt{h^2 + r^2})$$

using Lagrange multipliers. This gives us

$$\left\langle \frac{1}{3}2\pi r h, \frac{1}{3}\pi r^2 \right\rangle = \lambda \left\langle 2\pi r + \pi\sqrt{h^2+r^2} + \frac{\pi r^2}{\sqrt{h^2+r^2}}, \frac{\pi r h}{\sqrt{h^2+r^2}} \right\rangle$$

The second coordinate tells us that

$$\begin{aligned} r^2 =& \frac{3\lambda h r}{\sqrt{h^2+r^2}} \\ r =& \frac{3\lambda h}{\sqrt{h^2+r^2}} \\ \lambda =& \frac{r\sqrt{h^2+r^2}}{3h} \end{aligned}$$

Substituting this into the first coordinate

$$\begin{aligned} 2rh =& 3\lambda\left(2r + \sqrt{h^2+r^2} + \frac{r^2}{\sqrt{h^2+r^2}}\right) \\ 2rh =& 3\left(\frac{r\sqrt{h^2+r^2}}{3h}\right)\left(2r + \sqrt{h^2+r^2} + \frac{r^2}{\sqrt{h^2+r^2}}\right) \\ 2rh =& \frac{r}{h}\left(2r\sqrt{h^2+r^2} + rh^2 + r^3 + r^3\right) \\ 2rh =& \frac{r}{h}\left(h^2 2 + r(r + \sqrt{h^2+r^2})\right) \\ 2h^2 =& \left(h^2 + 2r(r + \sqrt{h^2+r^2})\right) \\ h^2 =& 2r(r + \sqrt{h^2+r^2}) \\ \frac{h^2}{2r} =& r + \sqrt{h^2+r^2} \end{aligned}$$

Substituting back into our constraint

$$1 = SA(r,h) = \pi r\left(\frac{h^2}{2r}\right)$$

which tells us that $h = \sqrt{2/\pi}$.

3. Write definition for a function $f(x,y)$ to be differentiable at a point p. Then show by any means that the function

$$f(x,y) = \frac{3x^2y - y^2}{x^2 + y^2}$$

is not differentiable.

Solution: A function $f(x,y)$ is called *differentiable* at a point (x_0, y_0) if

$$f(x_0 + \delta_x, y_0 + \delta_y) = f(x_0, y_0) + \delta_x f_x(x_0, y_0) + \delta_y f_y(x_0, y_0) + \epsilon(\delta_x, \delta_y)$$

where $\epsilon(\delta_x, \delta_y) < |\delta_x \delta_y|$ as $\delta_x, \delta_y \to 0$.

There are several different ways to show that this is not differentiable. One way is to notice that

$$f_x(x,y) = \frac{(x^2 + y^2)6x - (3x^2y - y^2)(2x)}{(x^2 + y^2)^2} = \frac{(6 - 3y)x^3 - (4y^2)x}{(x^2 + y^2)^2}$$

this function is not continuous in x and y at the origin. Taking $x = y$ gives a limit value of $-7/4$, but letting $x = 2y$ gives a limit value of $-22/25$. Since differentiable functions have continuous partial derivatives, this function is not differentiable.

4. A right cone of height 100, with vertex at the origin, is intersected with a sphere of radius 1, which is centered at the origin. What is the volume of the intersection? **Solution:** This is the region that in spherical coordinates is given by

$$\begin{aligned} 0 \leq \theta \leq 2\pi \\ 0 \leq \rho \leq 1 \\ 0 \leq \phi \leq \pi/4 \end{aligned}$$

Integrating

$$\iiint_D (1)\rho^2 \sin\phi d\rho d\phi d\theta$$

gives the volume.

5. What is the surface area of $1 + 3x + 2y^2$ over the triangle with corners (0,0), (0,1), (2, 1)?

Solution: The surface area infinitesimal is given by

$$dS = \sqrt{1 + f_x^2 + f_y^2} dA = \sqrt{1 + 9 + 4y^2} dA$$

The bounds of integration work out to

$$\int_0^1 \int_0^{y/2} \sqrt{10 + 4y^2} dx dy$$

6. The tastiness density of a unit orange is given by a function $T(x, y, z)$. A half orange eighth is given by this drawn region below:
   - Set up 3 integrals in Cartesian Coordinates that compute the tastiness of the half orange eighth.
   - Set up 3 integrals in Cylindrical Coordinates that compute the tastiness of the half orange eighth.
   - Set up 3 integrals in Spherical coordinates that compute the tastiness of the half orange eighth.