QUIZ 6

NAME:

Derivatives of Many Kinds: In this problem, f and $g : \mathbb{R}^2 \to \mathbb{R}$ are functions which are differentiable everywhere.

(a) Suppose you know that at the origin $\nabla f(0,0) = \langle .5, .2\rangle$.
What is $\partial_{\vec{v}} f(0,0)$, where $\vec{v} = \frac{1}{\sqrt{2}}\langle 1, 1\rangle$.

Solution: Use that $\partial_{\vec{v}} f = \nabla f \cdot \vec{v}$ to get that it is $7/(10\sqrt{2})$.

(b) Let $\vec{v} = \langle \sin(30), \cos(30)\rangle$, and let $\vec{w} = \langle 1, 0\rangle$. Suppose we know that at the origin,

$$\partial_{\vec{v}} g(0,0) = 2$$
$$\partial_{\vec{w}} g(0,0) = 3$$

What vector at the origin points in the direction of maximal increase?
Solution: We need to compute the gradient at the origin. Unfortunately, all we have right now is the $\partial_x = \partial_w$. However, we know that $\vec{v} = \langle 1/2, \sqrt{3}/2\rangle$ and therefore we can compute

$$\partial_{\vec{v}} f = 1/2\partial_x + \sqrt{3}/2\partial y$$

which gives us that $\partial y f(0,0) = (2/\sqrt{3})(3-1) = \frac{3}{\sqrt{3}}$. This gives us

$$\nabla f(0,0) = \langle 3, \frac{3}{\sqrt{3}}\rangle.$$

Maximum and Minimum. What is the maximum and minimum of the function $x^3 - 2xy^2$ on the region $x^2 + y^2 \leq 1$?
Solution: This is a max-min and Lagrange multipliers problem. Let's solve the max-min portion first.

$$\nabla f = \langle 3x + y^2, 4xy\rangle$$

This takes on a value of 0 only when $x = y = 0$. So this is a critical point that we need to check.
Along the boundary, can make a substitution into

$$f(x, \sqrt{1-x^2}) = x^3 - 2x(1-x^2)$$
$$= x^3 - 2x$$

This has derivative $f'(x) = 3x^2 - 2$ which has roots at $\pm\sqrt{\frac{2}{3}}$. The other possible points where a maxima might occur is when $x = \pm 1$. At these points, the maximal value of f is achieved, which are 1 and -1.

Lagrange Multipliers. Find the critical points and maximal/minimal values of $f(x, y, z) = x^2 - y^2$ on the surface $x^2 + 2y^2 + 3z^2 = 1$.
Solution: This is a Lagrange Multipliers problem. We have

$$\nabla f = \langle 2x, -2y, 0 \rangle$$
$$\nabla g = \langle 2x, 4y, 6z \rangle$$

Solving $\lambda \nabla g = \nabla f$ brings us into 3 cases :

- If $\lambda = 0$, $x = y = 0$, and z is allowed to range freely. This gives us $z = (\pm\sqrt{3})/3$.
- If $\lambda = 1$, then $z = y = 0$, and x is allowed to range freely. This gives us two points,
$$\langle \pm 1, 0, 0 \rangle$$
- If $\lambda = -\frac{1}{2}$, then $x = z = 0$, and y is allowed to range freely. This gives us two points,
$$\langle 0, (\pm\sqrt{2})/2, 0 \rangle$$

The minimal value is therefore $-1/2$, and the maximal value is 1.
The maximal value occurs when $x = \pm 1$.

Interesting Puzzle, Will not be graded. There is a pirate ship commanded by a just, communal, and mathematically minded pirate captain. Though she is a fearsome pirate, she lives by the pirate code when distributing booty to her mates. The pirate code states:

> The Pirate captain shall come up with a plan to distribute booty. That plan then goes to the crew for a vote. If the vote is in favor, the plan passes and booty is distributed. The pirates then go off to plunder more treasure. If the vote fails to pass or is a tie, then the captain shall be forced to walk the plank and jump into shark infested waters. The first mate shall become the captain, the second mate shall be promoted to first mate, and so on (the $(n + 1)$-mate will be promoted to n-mate.) The new captain will come up with a plan, and the process above repeated until booty has been distributed.

The pirate captain knows her 50 crew members well. In particular, she knows that all pirates vote on the following preferences:

- Pirates look out for their own neck before thinking about plunder— that is, no pirate will propose a plan to distribute booty that will cause them to walk the plank if they can afford to.
- Pirates prefer more treasure to less treasure, and don't really care about friendships. They don't make promises beyond following the pirate code.
- If there are two options before a pirate which reward equal amounts of treasure, then they will take the one that yields the most plank walkings (as they enjoy entertainment).

One day, the pirates come across 100 gold coins to distribute. What is the final distribution of gold coins among the pirates?