Problem 44

The problem asks to find the length of the curve

$$x(t) = 3\cos(t) - \cos(3t)$$
$$y(t) = 3\sin(t) - \sin(3t)$$

where t takes a value between 0 and π. We start by computing the infinitesimal length element

$$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

Let's do this computation.

$$\frac{dx}{dt} = -3\sin t + 3\sin(3t)$$
$$\frac{dy}{dt} = 3\cos t - 3\cos(3t)$$

Then we have that

$$\frac{ds}{dt} = \sqrt{9(\sin t - 3\sin(3t)^2 + 9(\cos t - \cos(3t))^2}$$
$$= 3\sqrt{(\sin^2 t + \cos^2 t) + (\sin^2(3t) + \cos^2(3t)) - 2(\sin(t)\sin(3t) + \cos(t)\cos(3t))}$$
$$= 3\sqrt{(2 - 2(\sin(t)\sin(3t) + \cos(t)\cos(3t))}$$

The term on the right is the difference of angle formula for cosine

$$= 3\sqrt{2 - 2\cos(3t - t)}$$
$$= 3\sqrt{2(1 - \cos(2t))}$$
$$= 3\sqrt{4\sin^2(t)}$$
$$= 6\sin(t)$$

Now integrating the length is easy

$$\text{Length} = \int_0^\pi \frac{ds}{dt} dt$$
$$= \int_0^\pi 6\sin(t)$$
$$= 12$$