These are a few review problems that I stole from previous midterms and finals at UC Berkeley, with partial (or starts at least) of solutions.

Problem 1. Find the general solution to

$$x + yy'e^{-x} = 0$$

Solution: This is a seperable equation, and so we can seperate it and then integrate. Starting with

$$\begin{aligned} x + yy'e^{-x} =& 0 \\ y\frac{dy}{dx}e^{-x} =& -x \\ ydy =& -xe^{x}dx \\ \int ydy =& -\int xe^{x}dx \\ y = \sqrt{-2\int xe^{x}dx} \end{aligned}$$

Problem 2. Find the general solution to

$$y'' + y = \sec x$$

Solution: We can solve using that terrible method the thing with undetermined coeffecients. We have two solutions to the auxiliarry equation are $y_1 = \sin x$ and $y_2 = \cos x$. Therefore the Wronskian $W(\sin x, \cos x) = 1$, which makes things a little bit easier. Then

$$u_1 = -\int \sec x \cos x dx \qquad u_2 = \int \sec x \sin x$$

Solving the left integral gives you $u_1 = -x$. The right integral is $u_2 = -\ln(\cos x)$. So the particular solution is $y_p = x\sin x + -(\ln(\cos x))\cos x$.

Problem 3. Does this following integral converge or diverge

$$\int_1^2 \frac{4w}{\sqrt[3]{w^2-4}}$$

Solution: Making the subsitution $u = w^2 - 4$, we have $\int_{-1}^{0} \frac{2}{\sqrt[3]{u}}$, which converges.

Problem 4. Find the solution to the differential equation $y'' - xy' - y = 0$ with initial conditions of $y(0) = 0$, $y'(0) = 1$.
Solution: This one is no fun, because you have to solve it by guessing a series. Guess the series $y = \sum_{n=0}^{\infty} a_n x^n$. Then

$$y'' = \sum_{n=2}^{\infty} a_n(n)(n-1)x^{n-2}$$

$$y = \sum_{n=1}^{\infty} a_n(n)x^{n-1}$$

Making the substitutions for indexes, we get

$$y'' = \sum_{n=0}^{\infty} a_{n+2}(n+1)(n)x^n$$

Plugging it back into our differential equation we have

$$\sum_{n=0}^{\infty}(a_{n+2}(n+1)(n)x^n) - x\sum_{n=1}^{\infty}(a_n(n)x^{n-1}) - \sum_{n=0}^{\infty}(a_nx^n) = 0$$

$$2\cdot 1a_2\cdot x^0 - a_0x^0 + \sum_{n=1}^{\infty}(a_{n+2}(n+1)(n) - na_n - a_n)x^n = 0$$

From here you just need to solve the relation for the a_n. We have for all n greater than 1 the relation

$$a_{n+2}(n+1)(n) - a_n(n+1) = 0$$

so you just plug in until you can find the pattern!

Problem 5. Determine if the following series converge or diverges. Justify your solution.

$$\sum_{n=1}^{\infty}\left(\frac{2n+2}{n+4}\right)^n$$

$$\sum_{n=1}^{\infty}\frac{n^n}{(n^2)!}$$

$$\sum_{n=1}^{\infty}(-1)^n\ln(1+1/n)$$

Solution: Here are two solutions to the second one. The first one (Chen Yusu) used the ratio test.

$$\lim_{n\to\infty}\frac{\frac{(n+1)^{n+1}}{((n+1)^2)!}}{\frac{n^n}{(n^2)!}} = \frac{\frac{(n+1)(n+1)^n}{(1\cdot 2\cdot 3\cdots(n^2-1)\cdot n^2\cdot(n^2+1)\cdots((n+1)^2-1)\cdot(n+1)^2}}{\frac{n^n}{(1\cdot 2\cdot 3\cdots(n^2-1)\cdot n^2}}$$

Combining the $(n+1)^n$ and the n^n

$$=(n+1)\left(1+\frac{1}{n}\right)^n\left(\frac{(1\cdot 2\cdot 3\cdots(n^2-1)\cdot n^2}{(1\cdot 2\cdot 3\cdots(n^2-1)\cdot n^2\cdot(n^2+1)\cdots((n+1)^2-1)\cdot(n+1)^2}\right)$$

$$=(n+1)\left(1+\frac{1}{n}\right)^n\left(\frac{1}{(n^2+1)\cdots((n+1)^2-1)\cdot(n+1)^2}\right)$$

Using the fact that $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$

$$=(n+1)e\left(\frac{1}{(n^2+1)\cdots((n+1)^2-1)\cdot(n+1)^2}\right) = 0$$

so by the ratio test it converges. Another way to solve this problem (Margaret Xiao) is by using the comparison test. Notice that whenever $n > 2$ we have that each term looks like

$$\frac{n\cdot n\cdot n\cdots n\cdot n}{1\cdot 2\cdots n\cdot(n+1)\cdot(n+2)\cdots(n^2-1)\cdot(n^2)} \leq \frac{1}{n!}$$

Where does this inequality come from. Well, the terms in the denominator coming after n are all greater than n, and there are at least n of them, so we can cancel them out with the ones that are in the numerator. Since $\sum\frac{1}{n!}$ converges, so does the above sum.

Problem 6. Determine the interval of convergence for the power series $\sum_{n=1}^{\infty}\ln(n)x^n$
Solution: Use the ratio test.

Problem 7. Calculate the volume of a sphere, using any method you wish.
Solution: Calculate the volume of rotation of the curve $y=\sqrt{1-x^2}$ for instance.