These are a few review problems that I stole from previous midterms at UC Berkeley.

1. Problem 1

Find the sums of the following series

(a) $\sum_{n=1}^{\infty}\left[\left(\ln\frac{n+1}{n}\right)-\ln\left(\frac{n+2}{n+1}\right)\right]$

(b) $\sum_{n=3}^{\infty}\left(3\left(\frac{3}{4}\right)^n-4\left(-\frac{1}{2}\right)^{n+1}\right)$

Solution: The first one is a telescoping series, so you just need to check that terms go to zero, and then the sum will be equal to the first term.
In the second problem, split the series into two smaller series, that is

$$\sum_{n=3}^{\infty}\left(3\left(\frac{3}{4}\right)^n-4\left(-\frac{1}{2}\right)^{n+1}\right)=\sum_{n=3}^{\infty}\left(3\left(\frac{3}{4}\right)^n\right)-4\sum_{n=3}^{\infty}\left(\left(-\frac{1}{2}\right)^{n+1}\right)$$

These are both geometric series, so you can find there sum. Be careful with, because the series start the sum at $n=3$, not $n=0$

2. Problem 2

Find the Maclaurin Series for $f(x)=xe^x$. Approximate $f(1)$ using a third degree approximation, and use Taylor's inequality to bound the error of your approximation.
Solution: You can show that the nth derivative of $f(x)$ is

$$f^{(n)}=ne^x+xe^x$$

Evaluating at $x=0$, we have that the Maclaurin series is

$$x+\frac{x^2}{1!}+\frac{x^3}{2!}+\frac{x^4}{3!}+\cdots+\frac{x^n}{(n-1)!}+\cdots$$

Using the third degree approximation gives $1+1+\frac{1}{2}$. Taylor's inequality says that the error should be bounded by the maximum of the fourth derivative on the interval $[-1,1]$. The maximum of $f^{(4)}(x)=4e^x+xe^x$ on the interval $[-1,1]$ occurs on the right endpoint. The value of the fourth derivative here is $f^{(4)}(1)=5e$. Letting $M=5e$, we have that the error for this approximation is less than $\frac{M}{4!}(x-a)^4=\frac{5e}{4!}$.

3. Problem 3

Solve the differential equation $y'\sqrt{x^2-2x+2}=y^2$ for the initial condition $y(0)=1$

4. Problem 4

Let $P'=P(P-1)^2$. Find the equilibria solutions for this differential equation, and the behavior of $\lim_{t\to\infty}P(t)$ for all initial values for $P(t)$.
Solution: The equilibria solutions to this differential equation are where $P'=0$, which occurs when $P=0$ and $P=1$. Since P' is non-negative, it means that P will always be increasing.

(1) Case 1: $P(0)\leq 0$. Then P will increase to 0 and stop, so $\lim_{t\to\infty}P(t)=0$.
(2) Case 2: $0<P(0)\leq 1$. Then P will increase to 1 and stop, so $\lim_{t\to\infty}P(t)=1$.
(3) Case 3: $1<P(0)$. Then P will keep on increasing forever and ever, so $\lim_{t\to\infty}P(t)=\infty$.

5. Problem 5

Jeff is trying to purify some salt water. He has a tub that holds 1000 liters of brine. He has a water filter that removes half of the salt content that passes through the filter. The tub has a hose that feeds into the water filter, which feeds back into the tub. Each day, 100 liters of water pass through the filter. Supposing that he starts with 10 Kg of salt in his water, what is amount of salt in the tub as a function of time?

Solution: Let Y be the amount of salt in the tank. The difficult part of this problem is setting up the differential equation. We have that $Y' = \text{rate in} - \text{rate out}$, and that $\text{rate in} = 1/2 \text{ rate out}$ (This is because the filter removes 1/2 the salt that passes through it.) The rate out is equal to

$$\text{Rate Out} = \text{Concentration of Brine} \cdot \text{Amount of Brine Leaving tank} = 100\frac{Y}{1000}$$

Therefore,

$$\begin{aligned} Y' =& \frac{1}{2}100\frac{Y}{1000} - 100\frac{Y}{1000} \\ =& -\frac{-Y}{20} \end{aligned}$$