Quiz 9

Name:

Problem 1. The general solution to a differential equation of the form

$$f''(x) + 2f'(x) + f(x) = 0$$

are solutions of the form $c_1e^{-x} + c_2xe^{-x}$

Verify that these are solutions to the differential equation, and find a solution that has $f(0) = 0$, and $f'(0) = 1$.

Solution: To check that $f(x) = c_1e^{-x} + c_2xe^{-x}$ is a valid solution to the differential equation, we take two derivatives

$$\begin{aligned} f(x) =& c_1e^{-x} + c_2xe^{-x} \\ f'(x) =& - c_1e^{-x} - c_2xe^{-x} + c_2e^{-x} \\ f''(x) =& c_1e^{-x} + c_2xe^{-x} - 2c_2e^{-x} \end{aligned}$$

Plugging in f, f', f'' into the differential equation shows that it does in fact satisfy the required conditions. Now to find the solution that fits our initial conditions. Letting $x = 0$, we have that

$$0 = f(0) = c_1e^0 + c_2(0)e^0$$

This shows that $c_1 = 0$. We then have

$$1 = f'(0) = -(0)e^0 - c_2(0)e^{-(0)} + c_2e^{-(0)}$$

Which shows that $c_2 = 1$

Problem 2. Consider the differential equation $y' = y(y-1)^2$. What are the equilibrium solutions to this differential equation, and for what values of c does a solution with initial values $y(0) = c$ have $\lim_{t\to\infty} y(t)$ be convergent? (it might be helpful to draw some pictures.)

Solution: We have that the equilibrium solutions are where $y' = 0$. So the equilibrium solutions are $y(t) = 1$ and $y(t) = 0$. We have that

$$y' \begin{cases} \geq 0 & \text{When } y \geq 0 \\ \leq 0 & \text{When } y < 0 \end{cases}$$

This means that if we start at a value greater than 1, the function will go to infinity. Likewise, if we start at a value less than 1, the function will go to negative infinity. If we start in the interval $(0, 1]$, we have the function will increase to 1. Finally, if we start at 0, the solution is stable. Therefore, the solutions that do not diverge as t goes to infinity have initial values $y(0)$ in $[0, 1]$

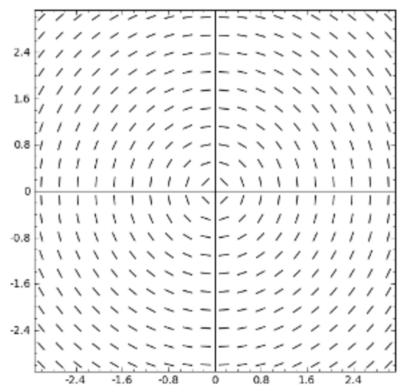
Problem 3. (15 Points) Consider the differenial equation $y'(x) = \frac{-x}{y(x)}$.

(1) Find all the points where the slope field has slope 0
(2) Find an equation for where the slope field has slope m (an isocline for slope m)

(10 points) Using these two pices of data, draw a slope field for the differential equation above with at least 8 slopes.

(5 points) Using any method you know, find a solution to the differential equation $y'(x) = \frac{-x}{y(x)}$.
(Hint: if you have a well drawn slope field, you should be able to guess the solution)

Solution: The isoclines for slope m are the lines $y = \frac{-1}{m}x$. If you plot out a couple of slopes, you should get a figure that looks like this



From here, you could guess that the solution was a circle. Plugging in $y = \sqrt{1-x^2}$ into the differential equation shows this to be a valud solution.