QUIZ 8

NAME:

Problem 1. Find the 3rd order approximation of $\tan(x)$ by a Taylor series. (Do *not* solve for the entire Taylor series, this is very hard!)

Solution: The easiest way to solve this problem was to take derivatives of tangent and then evaluate them at zero. We have

$$\begin{aligned} f(x) &= \tan(x) \\ f'(x) &= \sec^2(x) \\ f''(x) &= 2\tan(x)\sec^2(x) \\ f'''(x) &= 2\sec^2(x)\sec^2(x) + 2\tan(x)2\sec(x)\sec(x)\tan(x) \end{aligned}$$

Evaluating at zero gives

$$\begin{aligned} f(0) &= 0 \\ f'(0) &= 1 \\ f''(0) &= 0 \\ f'''(0) &= 2 \end{aligned}$$

So the taylor expansion to the third term is

$$\frac{x}{1!} + \frac{2x^3}{3!} + \cdots$$

Problem 2. Write a sum that approximates $\sin(1)$ to better than .1 and prove that your estimate is at least this accurate.

Solution: You can derive the Taylor series for $\sin(x)$, or you can have it memorized

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$$

We notice that this is an alternating series, so we may use the alternating series bound for error. We notice that $1/5! < .1$, so

$$1 - \frac{1^3}{3!} = 5/6$$

is an approximation of $\sin(1)$ that is better than .1

Problem 3. Write a sum that gives the 10th derivative of $e^x \frac{1}{1-x}$ at 0. (Hint: Use the product of power series)

Solution: When we take the product of these two series, we have that

$$\begin{aligned} e^x \frac{1}{1-x} &= \left(\sum_{n=0}^{\infty} \frac{x^n}{n!}\right)\left(\sum_{m=0}^{\infty} x^m\right) \\ &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} \frac{1}{k!} \cdot 1\right) x^n \end{aligned}$$

Letting $f(x) = e^x \frac{1}{1-x}$ and using the Taylor series expansion

$$= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

Looking at the tenth term of this series, we get

$$\frac{f^{(10)}(0)}{10!} x^{10} = \left(\sum_{k=0}^{10} \frac{1}{k!} \cdot 1\right) x^{10}$$

which tells us that $f^{(10)}(0) = 10! \left(\sum_{k=0}^{1} 0 \frac{1}{k!}\right)$