Here are a few review problems for power series.

Problem 1. Find a power series for $\frac{1}{4-4x+x^2}$

Solution: We first notice that the denominator of this series is a perfect square, so we can factor it. Then we will integrate to remove a power in the denominator, then make a substitution, and then do everything in reverse. All in all, it should look something like this;

$$
\begin{array}{ccc}
\frac{1}{4-4x+x^2} & & \\
\Big\downarrow \scriptstyle{factoring} & & \\
\frac{1}{(2-x)^2} & = & \left(\sum_{n=0}^{\infty} \frac{n}{2^{n+1}} x^{n-1}\right) \\
\Big\downarrow \scriptstyle{integrating} & & \scriptstyle{differentiating} \Big\uparrow \\
\frac{1}{2-x} & = & \sum_{n=0}^{\infty} \frac{1}{2}^{n+1} x^n \\
\Big\downarrow \scriptstyle{substitution} & & \Big\uparrow \\
\frac{1}{2-2u} & = & \frac{1}{2} \sum_{n=0}^{\infty} (\frac{x}{2})^n \\
\Big\downarrow & & \Big\uparrow \\
\left(\frac{1}{2}\right) \frac{1}{1-u} & = & \frac{1}{2} \sum_{n=0}^{\infty} u^n
\end{array}
$$

Problem 2. Use a power series to approximate this integral $\int_0^1 \ln(1+x^2)$ to better than 2 decimal places. Prove that you have approximated it to this accuracy.
Solution: We first need to find a power series for $\int_0^t \ln(1+x^2)dx$. We notice if we take two derivatives of this, we will have $\frac{1}{1+x^2}$, into which we can make a substitution $u = -x^2$.

$$
\begin{array}{ccc}
\int_0^t \ln(1+x^2)dx & = & C_0 + xC_1 + \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n+2}}{(2n+1)(2n+2)} \\
\Big\downarrow \scriptstyle{differentiating} & & \scriptstyle{integrating} \Big\uparrow \\
\ln(1+t^2) & = & C_1 + \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n+1}}{2n+1} \\
\Big\downarrow \scriptstyle{differentiating} & & \scriptstyle{integrating} \Big\uparrow \\
\frac{1}{1+t^2} & = & \sum_{n=0}^{\infty} (-1)^n t^{2n} \\
\Big\downarrow \scriptstyle{substitution} & & \Big\uparrow \\
\frac{1}{1-u} & = & \sum_{n=0}^{\infty} u^n
\end{array}
$$

As the sum is an alternating series, we can use the alternating series test to estimate what this sum is.

Problem 3. Find a power series for the product $\frac{1}{1-x}\arctan(x)$.
Solution: We can use the rule for the product of the power series to get a power series for each $\frac{1}{1-x}$ and $\arctan(x)$. Notice that the derivative of $\arctan(x)$ is $\frac{1}{1+x^2}$, which looks suspiciously like something we should get a power series for. (I'm not going to actually compute this power series, but what you get is $\arctan(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$. We know that $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^m$. The power series for $\frac{1}{1-x}\arctan(x)$ is

$$\frac{1}{1-x}\arctan(x) = \left(\sum_{n=0}^{\infty} x^m\right)\left(\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{2n+1}\right)$$
$$=(1+x+x^2+x^3+\dots)(\frac{x}{1}-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\dots)$$

Taking the product of a few of these terms by hand gives us

$$=(1)x+(1)x^2+(1-\frac{1}{3})x^3+(1-\frac{1}{3})x^4+(1-\frac{1}{3}+\frac{1}{5})x^5+\dots$$

So the coefficient in front of x^m is the alternating sum of all odd fractions where the denominator is less than m.

Problem 4. Find the radius of convergence and interval of convergence for the power series $\sum_{n=1}^{\infty}\frac{n}{4^n}(x+1)^n$.
Solution: Use Ratio Test