Quiz 7

Name:

Problem 1. Find the center, radius and interval of convergence for this power series.

$$\sum_{n=0}^{\infty} n!(x-5)^n$$

Solution: We use the ratio test.

$$\lim_{n\to\infty} \frac{(n+1)!(x-5)^{n+1}}{(n)!(x-5)^n} = \lim_{n\to\infty}(n+1)(x-5)$$

This limit is finite only when $x = 5$. Therefore we see that this series converges only when $x = 5$. So the series is centered at 5, has a radius of convergence of 0, and an interval of convergence $[5, 5]$.

Problem 2. Find a power series that fits this function. Then find radius and center of convergence of this power series.

$$\frac{(x+2)^3}{1+4x+x^2}$$

Solution: Here is how I would solve the problem. Follow the arrows!

$$\frac{(x+2)^3}{1+4x+x^2} = \sum_{n=0}^{\infty} -3^{-(n+1)}(x+2)^{2n+3}$$

We pull out a term and start to complete the square ↓ / ↑ Bringing in the terms from the outside

$$(x+2)^3 \frac{1}{1-4+4+4x+x^2} = -\frac{(x+2)^3}{3}\sum_{n=0}^{\infty}\left(\frac{1}{3^n}\right)^n (x+2)^{2n}$$

Completing the square ↓ / ↑ Breaking up the power

$$(x+2)^3 \frac{1}{-3+(x+2)^2} = -\frac{(x+2)^3}{3}\sum_{n=0}^{\infty}\left(\frac{1}{3}(x+2)^2\right)^n$$

Substituting $3u = (x+2)^2$ ↓ / ↑ Undoing the substitution for u

$$-\frac{(x+2)^3}{3}\left(\frac{1}{1-u}\right) \longrightarrow = \longrightarrow -\frac{(x+2)^3}{3}\sum_{n=0}^{\infty} u^n$$

And now we just need to convert the sum $\sum_{n=0}^{\infty} -3^{-(n+1)}(x+2)^{2n+3}$ into a proper power series. Letting

$$c_m = \begin{cases} -(3)^{-\frac{m-1}{2}} & m \text{ is odd and greater than 2} \\ 0 & \text{Otherwise} \end{cases}$$

We have that $\sum_{m=0}^{\infty} c_m x^m$ is the desired power series. It is centered at $x = -2$, and has a radius of convergence of $\sqrt{3}$

Problem 3. Evaluate the following sum.

$$\frac{1}{4\cdot 1}+\frac{1}{16\cdot 2}+\frac{1}{64\cdot 3}+\frac{1}{256\cdot 4}+\ldots+\frac{1}{2^{2n}\cdot n}+\ldots$$

Solution: We notice that this sum can be written

$$\begin{array}{ccc}
\sum_{n=1}^{\infty}\frac{1}{2^{2n}\cdot n} & = & -\ln(3/4) \\
\text{Rearranging the terms}\downarrow & & \uparrow\text{Simplifying} \\
\sum_{n=1}^{\infty}\frac{(1/4)^n}{n} & = & -\ln(1-1/4) \\
\text{Letting } x=1/4\downarrow & & \uparrow\text{Solving for } C \text{ shows that } C=0.\text{ Putting the } x \text{ back in.} \\
\sum_{n=1}^{\infty}\frac{(x)^n}{n} & = & -\ln(1-x)+C \\
\text{Differentiating the series}\downarrow & & \uparrow\text{Now integrating to undo the differentiation} \\
\sum_{n=1}^{\infty}x^{n-1} & \xrightarrow{=} & \frac{1}{1-x}
\end{array}$$