These are a few review problems that I stole from previous midterms at UC Berkeley. I wrote up the solutions pretty late at night, so I have skipped quite a few steps in these solutions– but it is probably good practice to go and fill in the blanks that I've left here. I hope this helps! –Jeff

1. Problem 1

For what values of p is the series $\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right) n^p$ convergent?

Solution: If $p < 0$, the series converges. Use the limit comparison test with $\sum_{n=1}^{\infty} \frac{1}{n^{1-p}}$

2. Problem 2

For each of the following statements, tell whether it is true or false, and give a justification for your answer. In particular, the falsity of an "if/then" statement should be justified by a counterexample.

(1) If the sequence $\{a_n\}$ converges to L, then a_{100} is closer to L than a_{99} is.
(2) If the series $\sum_{n=1}^{\infty} a_n$ is convergent, then the sequence $\{a_n\}$ is also convergent.
(3) If $a_n \to 0$ as $n \to \infty$, then the series $\sum_{n=1}^{\infty} (-1)^n a_n$ is convergent.

Solution:

(1) No! Consider the series

$$a_n = \begin{cases} 0 & \text{If } n < 99 \\ 1 & \text{If } n = 100 \\ 0 & \text{If } n > 100 \end{cases}$$

(2) Yes! This is by the divergence test. if a_n didn't go to zero, then neither would $\sum a_n$, so if $\sum a_n$ converges, a_n must go to 0.
(3) No! Here is the easy example.

$$a_n = \frac{(-1)^n}{n}$$

(Notice here that the alternating series test requires a_n to be monotonically decreasing to zero, which was *not* a requirement listed in the problem)

3. Problem 3

$\int \frac{\ln(x^2)}{x^2} dx$

Solution: We can first simplify the problem a little bit by looking at $\int \frac{2\ln(x)}{x^2}$ instead. Then letting $u = 2\ln(x)$, $dv = \frac{1}{x^2}dx$, we have that $du = \frac{2}{x}$ and $v = \frac{-1}{x}$. Then we have

$$\begin{aligned} \int \frac{\ln(x^2)}{x^2} dx = & \frac{-2\ln(x)}{x} - \int \frac{-2}{x^2} dx \\ = & \frac{\ln(x^2)}{x^2} dx \\ = & \frac{-2\ln(x)}{x} - \frac{2}{x} + C \end{aligned}$$

4. Problem 4

Calculate $\int \frac{8-16x}{8x^2-4x+1} dx$

Solution: Someone in the class came up with this solution, and it was totally better than mine so we are going to go with it. Look at the derivative of the $8x^2 - 4x + 1$. The derivative of the bottom is $16x - 4$. Look suspiciously like the top, right? So you might try ans break the top into

$$\begin{aligned} \int \frac{8-16x}{8x^2-4x+1} dx &= \int \frac{4-16x}{8x^2-4x+1} + \frac{4}{8x^2-4x+1} dx \\ &= \int \frac{4-16x}{8x^2-4x+1} dx + \int \frac{4}{8x^2-4x+1} dx \end{aligned}$$

Using u sub on the left, and completing the square on the right

$$\begin{aligned} &= \int \frac{-du}{u} + \int \frac{4}{8(x-\frac{1}{4})^2+\frac{1}{2}}dx \\ &= -\ln(u) + \int \frac{8}{(4x-1)^2+1}dx \end{aligned}$$

Now using v sub on the right

$$\begin{aligned} &= -\ln(8x^2-4x+1) + 8\int \frac{1}{v^2+1}\frac{dv}{4} \\ &= -\ln(8x^2-4x+1) + 2\tan^{-1}(v) + C \\ &= -\ln(8x^2-4x+1) + 2\tan^{-1}(4x-1) + C \end{aligned}$$

5. Problem 5

Determine the length of $x = \frac{1}{2}y^2$ for $0 \leq x \leq \frac{1}{2}$. Assume that y is positive.
Solution: Use the formula for the length. We have that $\frac{d}{dy}x = y$ and $ds = \sqrt{1+(y)^2}dy$

$$\begin{aligned} \text{Length} &= \int ds \\ &= \int \sqrt{1+(y)^2}dy \end{aligned}$$

Make the substitution $y^2 = \tan\theta$

$$\begin{aligned} &= \int \sqrt{1+\tan^2\theta}\sec^2\theta d\theta \\ &= \int \sec^3\theta d\theta \end{aligned}$$

The dreaded $\int \sec^3\theta d\theta$ returns! I'm not going to work this one out, because there is a whole entire wikipedia article on how to integrate $\sec^3\theta$.